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Spin(7) Instantons and Hermitian Yang-Mills Connections for the Stenzel Metric

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0 Introduction

0.1 Instantons

Instantons are special connections characterized by the fact that they solve certain first order PDE systems that are stronger than the Yang-Mills equations. In fact, over a compact manifold without boundary, instantons are absolute minimizers of the Yang-Mills functional. Their study is largely motivated by the work of Donaldson on 4-dimensional smooth topology. Over a compact 4-manifold M, the space of 2-forms decomposes into the $+1$ and -1 eigenspaces of the Hodge star operator. The forms lying in the latter are termed *anti-self dual* (ASD). The connections whose curvature is ASD are termed *ASD instantons.* A simple argument involving Chern-Weil theory shows that they are precisely the absolute minimizes of the Yang-Mills functional and are thus Yang-Mills. Donaldson's study of the moduli space of ASD instantons led to his famous diagonalizability theorem for the intersection form on $H^2(M, \mathbb{Z})$. This result has many striking consequences such as the existence of uncountably many non-diffeomorphic smooth structures on $\mathbb{R}^4$ and the existence of non smoothable topological 4-folds.

The study of ASD instantons is also fundamental in Instanton-Floer homology. Let Y be a 3-manifold and P a principal bundle over Y. One attempts to do Morse theory for the Chern-Simons functional. Smooth paths in the space of connections can be thought of as connections on the pullback bundle $\pi^\star P$ over the cylinder $X = Y \times \mathbb{R}$ (where π is projection to the first factor). The gradient flowlines of the Chern-Simons functional correspond to ASD instantons in temporal gauge over X.

There is a conjectural analogue of the 3+1 dimensional Instanton-Floer theory in dimension 7+1 (Salamon [10] p.88). The 3-manifold Y is now replaced by a 7 dimensional manifold with holonomy in the exceptional group G_2. Such manifolds are known as *G_2 manifolds.* The cylinder inherits a metric with holonomy in the exceptional group Spin(7). Such manifolds are known as *Spin(7) manifolds.* The geometric structure of a Spin(7) manifold is captured by a globally defined 4-form, its *Cayley calibration.* This form allows us to write down an instanton equation for connections over X. Solutions are known as *Spin(7) instantons.* One introduces a G_2 analogue of the Chern-Simons functional on connections over Y. There is then a 1-1 correspondence between the gradient flowlines of this functional and Spin(7) instantons in temporal gauge. It is then apparent that understanding Spin(7) instantons is likely to have major consequences for geometry and topology.

0.2 Overview of the Present Report

The inclusion of SU(4) in Spin(7) allows one to endow a Calabi-Yau (CY) 4-fold with a natural Spin(7) structure. On such a manifold (in fact over any Kähler manifold), one may define yet another type of instanton: the *Hermitian Yang-Mills* (HYM) connections. A natural question is to ask whether these connections are related to the Spin(7) instantons associated to the induced Spin(7) structure. One immediately observes that HYM is a stronger condition. In the compact case, it is known that as long as a Hermitian Yang-Mills connection exists, the two types of instantons coincide (Lewis [5]). Consequently, if one hopes to display a compact counterexample to equivalence, there must not be any HYM connections at all. Furthermore, we have a general existence theorem for HYM connections over stable holomorphic bundles (Uhlenbeck, Yau [13]). This restricts the choices of bundles one could look at. Finally, compact special holonomy manifolds admit not continuous symmetries (Joyce, [2]). This precludes the usage of symmetry techniques. We are thus motivated to look for a non-compact counterexample. Since Lewis's argument is essentially an energy estimate, it does not apply to the noncompact setting.

In this report we study a non-compact cohomogeneity one CY 4-fold: the cotangent bundle of the 4-sphere equipped with the Stenzel metric (Stenzel [11]). We use the natural SO(5) action to reduce the instanton equations to tractable ODEs and proceed to study the SO(5)-invariant solutions. We treat the abelian case in detail. We prove that the two equations are equivalent away from the singular orbit S^4 and that they admit a unique smooth solution there. We give an explicit representation formula for this solution and use it to prove that the instanton breaks down near S^4. We thus obtain a nonexistence result. This study sets the ground for the analysis of more complicated structure groups in future work.

1 Spin(7) Instantons and Hermitian Yang-Mills Connections

In this section we collect background material on Spin(7)-manifolds, Spin(7) instantons and HYM connections. We do not give complete proofs and instead refer to the expository paper by Salamon-Walpuski (Salamon-Walpuski [10]) the book (Joyce [2]) and Lewis' foundational thesis (Lewis [5]). Our aim is to provide enough information to specify and motivate the object of our study.

We begin by introducing certain linear algebraic structures special to dimension 8. These structures provide the pointwise model for Spin(7) manifolds. We define Spin(7) manifolds and introduce the Spin(7) instanton equation. We show that it is stronger than the Yang-Mills equation and that, over compact manifolds, its solutions are precisely the absolute minimizers of the Yang-Mills functional. Finally, we introduce the HYM equations on a Calabi-Yau 4-fold. We prove that they are stronger than the Spin(7) instanton equations. We conclude the section by presenting Lewis' theorem on the relation of HYM connections and Spin(7) instantons in the compact case.

1.1 Octonionic Linear Algebra: Tripple Cross Products, Cayley Calibrations and the Group Spin(7)

A *normed algebra* consists of a fintie dimensional real Euclidean vector space $\big(W, \langle\cdot\rangle\big)$, equipped with a bilinear map:

$$\begin{gathered}\otimes^2 W \to W \\ (u,v) \mapsto uv\end{gathered} \tag{1.1}$$

such that

$$|uv| = |u||v| \tag{1.2}$$

and a unital element $1 \in W$ such that:

$$1w = w1 = w \text{ for all } w \in W. \tag{1.3}$$

One defines the real and imaginary parts of W as:

$$\mathfrak{Re}(W) \stackrel{\text{def}}{=} \mathrm{Span}_{\mathbb{R}}(1) \tag{1.4}$$

$$\mathfrak{Im}(W) \stackrel{\text{def}}{=} \mathfrak{Re}(W)^{\perp} \tag{1.5}$$

A *cross product* on a real finite dimensional Euclidean vector space $\big(V, \langle\cdot\rangle\big)$ is a bilinear operation:

$$\begin{gathered}\otimes^2 V \to V \\ (u,v) \mapsto u \times v\end{gathered} \tag{1.6}$$

such that:

$$\langle u \times v, u\rangle = \langle u \times v, v\rangle = 0 \tag{1.7}$$

$$|u \times v|^2 = |u|^2|v|^2 - \langle u, v\rangle^2 \tag{1.8}$$

The imaginary part of a normed algebra W carries a natural cross product defined as:

$$u \times v \stackrel{\text{def}}{=} uv + \langle u, v\rangle \tag{1.9}$$

One may obtain a complete classification of cross products using techniques of elementary linear algebra. Due to the above observation, this also gives a classification of normed algebras. Essentially, the only possibilities are the real numbers ($\mathbb{R}$), the complex numbers ($\mathbb{C}$), the quaternions ($\mathbb{H}$) and the octonions ($\mathbb{O}$). The cross products for the first two vanish. The cross product for $\mathbb{H}$ is the usual cross product on $\mathbb{R}^3$. The cross product on $\mathbb{O}$ is the usual cross product on $\mathbb{R}^7$.

One writes:

$$\mathbb{O} = \mathrm{Span}_{\mathbb{R}}(1, i, j, k, e, ei, ej, ek) \tag{1.10}$$

where the basis vectors are orthonormal, the elements i, j, k, e are anti-commuting with square 1 and $ij = k$. The standard cross product on $\mathbb{R}^7 \cong \mathfrak{Im}(\mathbb{O})$ is then given by the formula 1.9.

We introduce two more linear algebraic objects that will be of interest to us: *tripple cross products* and *Cayley Calibrations*. A *tripple cross product* on real finite-dimensional Euclidean vector space $\big(W, \langle\cdot\rangle\big)$ is an alternating trilinear map:

$$\otimes^3 W \to W \tag{1.11}$$

$$u \otimes v \otimes w \mapsto u \times v \times w \tag{1.12}$$

satisfying:

$$\langle u \times v \times w, u\rangle = \langle u \times v \times w, v\rangle = \langle u \times v \times w, w\rangle = 0 \tag{1.13}$$

$$|u \times v \times w| = |u \wedge v \wedge w| \tag{1.14}$$

An alternating 4-form

$$\Phi \in \Lambda^4 W^\star \tag{1.15}$$

is called a *Cayley calibration* for $(W, \langle\cdot,\cdot\rangle)$ if it is *nondegenerate* and *compatible with the inner product*. The first condition means that for all linearly independent $u, v, w \in W$ there exists some $x \in W$ such that $\Phi(u, v, w, x) \neq 0$. The second one is that the map defined by:

$$(u, v, w) \mapsto u \times v \times w$$

$$\langle x, u \times v \times w\rangle = \Phi(x, u, v, w) \tag{1.16}$$

is a triple cross product on W.

Cayley calibrations are automatically self-dual for the Hodge-star operator determined by the metric and the orientation:

$$\star_g \Phi = \Phi \tag{1.17}$$

It is immediate from the definitions that a triple cross product determines a Cayley calibration and vice versa. Explicitly, this is done using equation (1.16).

If W carries a tripple cross product and $e \in W$ is a unit vector, then the orthogonal complement:

$$V = e^\perp$$

caries a natural cross product defined by:

$$u \times_e v \overset{\text{def}}{=} u \times e \times v \tag{1.18}$$

Since cross product only occur in dimensions $0, 1, 3, 7$, tripple cross products and Cayley calibrations only exist in dimensions $0, 1, 2, 4, 8$. In the first three cases they are trivial.

The pointwise model for Spin(7) manifolds is the standard Cayley calibration Φ_{standard} on $\mathbb{R}^8 \cong \mathbb{O}$. Denoting the standard orthonormal basis of the octonions by $e_1, ..., e_8$ and its dual basis by $\epsilon^1, ..., \epsilon^8$, we have that (Salamon Walpuski **??** p.35):

$$\begin{aligned} \Phi_{\text{standard}} = \epsilon^{1234} - \epsilon^{1256} - \epsilon^{1278} - \epsilon^{1357} + \epsilon^{1368} - \epsilon^{1458} - \epsilon^{1467} \\ + \epsilon^{5678} - \epsilon^{3478} - \epsilon^{3456} - \epsilon^{2468} + \epsilon^{2457} - \epsilon^{2367} - \epsilon^{2358} \end{aligned} \tag{1.19}$$

We define the group Spin(7) to be the isotropy subgroup of $\Phi_{standard}$ in GL(8) i.e.:

$$\text{Spin}(7) \overset{\text{def}}{=} \big\{g \in \text{GL}(8) \text{ s.t. } g^\star\Phi_{\text{standard}} = \Phi_{\text{standard}}\big\} \tag{1.20}$$

All linear automorphisms that preserve Φ_{standard} also preserve the Euclidean metric. It is then the case that:

$$\text{Spin}(7) \subset \text{SO}(8) \tag{1.21}$$

In fact, it holds that:

$$\mathrm{Spin}(7) = \left\{g \in \mathrm{SO}(8) \text{ s.t. } gu \times gv \times gw = g(u \times v \times w)\right\} \tag{1.22}$$

where the operation $u \times v \times w$ is the tripple cross product associated to Φ_{standard}.

The group Spin(7) is a semisimple, connected, simply connected 21-dimensional Lie group.

The group SO(8) acts on $\mathfrak{so}(8)$ by the adjoint action. Since Spin(7)<SO(8), we have that $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$. The adjoint action of SO(8) can be restricted to Spin(7). The latter then preserves the 7-dimensional subspace $\mathfrak{spin}(7)^{\perp} \subset \mathfrak{so}(8)$. This gives rise to a map $\mathrm{Spin}(7) \to \mathrm{SO}(7)$. This map is a nontrivial double covering, exhibiting Spin(7) as the universal cover of SO(7). We thus recover the usual definition for spin groups.

1.2 Spin(7) Manifolds and Spin(7) Instantons

Let M be an 8-dimensional manifold. Consider the (nonlinear) subbundle of $\Lambda^4 T^\star M$ defined by:

$$\mathcal{A}M \overset{\text{def}}{=} \coprod_{p \in M} \mathcal{A}_p M \tag{1.23}$$

where

$$\mathcal{A}_p M \overset{\text{def}}{=} \left\{\omega \in \Lambda^4 T_p^\star M : \exists \text{ oriented linear isomorphism } \phi : T_p M \xrightarrow{\sim} \mathbb{R}^8 \text{ taking } \omega \text{ to } \Phi_{\text{standard}}\right\}$$

the standard fiber of this bundle is diffeomorphic to the 43 dimensional manifold $\mathrm{GL}_+(8)/\mathrm{Spin}(6)$ and is thus of codimension 27 in $\Lambda^4 T_p^\star M$ (Joyce [2] p.240). We have the following definition

Definition 1.1. A Spin(7) structure on an 8-manifold M is a smooth section:

$$\Phi \in C^\infty(\mathcal{A}M)$$

Note that since the bundle $\mathcal{A}M$ is not linear, it is not always the case that a Spin(7) structure exists.

Spin(7)-structures are in $1-1$ correspondence with reductions of the frame bundle $\mathrm{Fr}(TM)$ to Spin(7).

Since $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$, a Spin(7) structure induces a Riemannian metric on M. This is done pointwise by writing down the standard Euclidean metric in any of the frames inside the Spin(7) reduction of $\mathrm{Fr}(TM)$.

A Spin(7) structure is called *torsion free* if:

$$\nabla_g \Phi = 0 \tag{1.24}$$

where ∇_g is the Levi-Civita connection of the metric determined by Φ. In analogy to Kähler structures, a Spin(7) structure Φ is torsion free if and only if (Joyce [2] p.240):

$$d\Phi = 0 \tag{1.25}$$

We finally formulate the following definition:

Definition 1.2. A Spin(7) manifold is an 8-dimensional smooth manifold M equipped with a torsion free Spin(7) structure Φ.

We usually denote a Spin(7) manifold M as (M, Φ, g) in order to emphasize that Φ determines a Riemannian metric.

A Spin(7) manifold (M, Φ, g) satisfies:

$$\mathrm{Hol}\left(\nabla_g\right) \subseteq \mathrm{Spin}(7) \tag{1.26}$$

The following proposition holds (Joyce [2] prop. 11.4.5):

Proposition 1.3. *Let (M, g) be a Riemannian 8-manifold. Suppose that:*

$$Hol\left(\nabla_g\right) \subseteq Spin(7) \tag{1.27}$$

Then (M, g) is Ricci flat.

We conclude that Spin(7) manifolds are Ricci flat. They are thus of special interest to Physicists (Ricci flat metrics are Einstein vaccuum solutions).

Our task is to do gauge theory over Spin(7) manifolds. Since the curvature of a principal connection is an ad(P)-valued 2 form over M, we are interested in understanding the space $\Lambda^2 T^\star M$.

Let (M, Φ, g) be a Spin(7) manifold. The space of 2-forms on M splits into an orthogonal direct sum of irreducible Spin(7) representations (Lewis [5] p.9):

$$\Lambda^2 T^\star M = \Lambda^2_7 \oplus \Lambda^2_{21} \tag{1.28}$$

The subscripts denote the dimensions of the subrepresentations. We denote the associated orthogonal projection operators by:

$$\pi_7 : \Lambda^2 \twoheadrightarrow \Lambda^2_7 \tag{1.29}$$

and

$$\pi_{21} : \Lambda^2 \twoheadrightarrow \Lambda^2_{21}. \tag{1.30}$$

We have the following characterization of Λ^2_{21} (Salamon-Walpuski [10] p. 81):

$$\Lambda^2_{21} = \left\{\omega \in \Lambda^2 T^\star M \text{ s.t. } \star_g (\Phi \wedge \omega) = -\omega\right\} \tag{1.31}$$

We may rewrite the condition in (1.31) as:

$$\star_g \omega = -\Phi \wedge \omega \tag{1.32}$$

We observe that (1.32) resembles the ASD condition from Donaldson theory. This motivates the following definition:

Definition 1.4. Let (M, Φ, g) be a Spin(7) manifold. Let P be a principal G-bundle over M. A connection $A \in \mathcal{A}(P)$ is a Spin(7) instanton if:

$$F_A \in C^\infty\left(\Lambda^2_{21} \otimes \mathrm{ad}(P)\right) \tag{1.33}$$

Clearly, an equivalent characterization is that:

$$\star_g F_A = -\Phi \wedge F_A \tag{1.34}$$

We have the following immediate observation:

Proposition 1.5. *Spin(7) instantons are Yang-Mills.*

Proof. This is a simple calculation reminiscent of the corresponding calculation for the ASD case:

$$\begin{aligned} d_A^\star F_A &= - \star_g d_A \star_g F_A \\ &= - \star_g d_A \left(-\Phi \wedge F_A\right) \\ &= \star_g \left(d\Phi \wedge F_A + \Phi \wedge d_A F_A\right) \\ &= 0 \end{aligned} \tag{1.35}$$

In the final step (1.35) we used the torsion freeness of the Spin(7) structure and the differential Bianchi identity:

$$d_A F_A = 0 \tag{1.36}$$

□

On a compact manifold without boundary, proposition (1.5) can be significantly strengthened (Lewis [5] prop. 3.1):

Theorem 1.6. *Let (M, Φ, g) be a compact Spin(7) manifold without boundary. Let P be a principal G bundle over M, where $G \subseteq GL(r, \mathbb{C})$. Let $A \in \mathcal{A}(P)$. We have that:*

$$\mathcal{YM}(A) = \mathcal{Q}(P, \Phi) + 4 \int_M |\pi_7 F_A|^2 \, dV_g \tag{1.37}$$

where $\mathcal{Q}(P, \Phi)$ is a quantity independent of A and determined only by the Spin(7) structure Φ and the topology of the bundle. In particular:

$$\mathcal{Q}(P, \Omega) = 4\pi^2 \int_M \left[2c_2(P) - c_1^2(P)\right] \wedge [\Phi] \tag{1.38}$$

Proof. (Sketch) The full proof can be found in (Lewis [5] p.22). One defines:

$$Q(A) \stackrel{\text{def}}{=} \int_M |\pi_{21} F_A|^2 \mathrm{dV}_g - 3 \int_M |\pi_7 F_A|^2 \mathrm{dV}_g \tag{1.39}$$

The task is then to prove that for any $A \in \mathcal{A}(P)$ we have:

$$Q(A) = \mathcal{Q}(P, \Phi) \tag{1.40}$$

This is done by direct calculation. □

Corollary 1.7. *Let (M, Ω, g) be a compact Spin(7) manifold without boundary. Let P be a principal G bundle over M, where $G \subseteq GL(r, \mathbb{C})$. Suppose that there exists a Spin(7) instanton. The Spin(7) instantons are then precisely the absolute minimizers of the Yang-Mills fuinctional.*

Proof. The Yang-Mills energy of a connection A is given by the formula (1.37). The Spin(7) instantons are precisely the connections for which the second term vanishes. As such, if A is any connection and $A_{Spin(7)}$ is a Spin(7) instanton, we have:

$$\mathcal{YM}\left(A_{Spin(7)}\right) \leq \mathcal{YM}(A) \tag{1.41}$$

□

In fact, it follows that the Spin(7) instantons all share the same minimal Yang-Mills energy equal to:

$$\mathcal{YM}_{Spin(7)} = \mathcal{Q}(P, \Omega) \tag{1.42}$$

1.3 Spin(7) Instantons on Calabi-Yau 4-Folds and Relation to HYM Connections in the Compact Case

Let (M, J, ω, Ω) be a Calabi-Yau 4-fold. Here J denotes the complex structure, ω denotes the Kähler form and Ω denotes the holomorphic volume form. The inclusion SU(4) ⊂ Spin(7) allows us to write down a natural Spin(7) structure on M. In terms of the data (J, ω, Ω) this takes the form (Salamon-Walpuski [10] p.81):

$$\Phi = \frac{\omega^2}{2} + \mathfrak{Re}(\Omega) \tag{1.43}$$

A trivial calculation shows that Φ is torsion free. The metric induced by the Cayley Calibration Φ agrees with the Calabi-Yau metric induced from ω and J (Joyce [2] prop. 11.4.11).

The spaces Λ^2_7 Λ^2_{21} decompose further into orthogonal SU(4) irreducible components:

$$\Lambda^2_{21} = \Lambda^{1,1}_0 \oplus B \tag{1.44}$$

$$\Lambda^2_7 = \langle \omega \rangle \oplus C \tag{1.45}$$

where B and C are 6 dimensional real subspaces of $\mathfrak{Re}\left(\Lambda^{2,0} \oplus \Lambda^{0,2}\right)$ and $\Lambda^{1,1}_0$ is the 15-dimensional subspace of $\mathfrak{Re}\left(\Lambda^{1,1}\right)$ defined as the orthogonal complement of the Kähler form.

Hermitian Yang-Mills (HYM) connections are instantons defined using this splitting.

Definition 1.8. Let (M, J, ω, Ω) be a Calabi-Yau 4-fold. Let P be a principal G-bundle over M. A connection $A \in \mathcal{A}(P)$ is Hermitian Yang-Mills if:

$$F_A \in C^\infty\left(\Lambda_0^{1,1} \otimes \mathrm{ad}(P)\right) \tag{1.46}$$

It is clear that the HYM condition can be written as a system of PDE for the connection (Li [6]):

$$F_A \wedge \star\omega = 0 \tag{1.47}$$

$$F_A^{2,0} = F_A^{0,2} = 0 \tag{1.48}$$

Since F_A is a real form, if its $(0,2)$ part vanishes, so does its $(2,0)$ part. Since Ω is of bidegree $(4,0)$, we have that:

$$F_A \wedge \Omega = F_A^{0,2} \wedge \Omega \tag{1.49}$$

and furthermore:

$$F_A^{0,2} \wedge \Omega = 0 \iff F_A^{0,2} = 0 \tag{1.50}$$

From these remarks it follows that:

$$F_A^{2,0} = F_A^{0,2} = 0 \iff F_A \wedge \Omega = 0 \tag{1.51}$$

Finally, we see that a connection is HYM if and only if:

$$F_A \wedge \star\omega = 0 \tag{1.52}$$

$$F_A \wedge \Omega = 0 \tag{1.53}$$

Since $\Lambda_0^{1,1} \subset \Lambda_{21}^2$, it follows that an HYM connection is automatically a Spin(7) instanton. Consequently it is Yang-Mills (justifying the terminology). In fact, over a compact manifold with no boundary, as soon as an HYM connection exists, the Spin(7) instantons and the HYM connections coincide, they are precisely the absolute minima of the Yang-Mills functional and they share the same minimal energy. The following theorem is due to Lewis (Lewis [5] Thm. 3.1).

Theorem 1.9. *Let (M, J, ω, Ω) be a Calabi-Yau 4-fold. Let P be a principal G-bundle over M such that $G \subseteq GL_r(\mathbb{C})$. Suppose that an HYM connection exists. Then any Spin(7) instanton is HYM.*

Proof. (Sketch) A complete proof can be found in ([5]). The argument is similar to the proof of theorem 1.6. The role of the Cayley calibration is played by the real part of the holomorphic volume. In particular, one proves that the quantity:

$$Q(A) \overset{\text{def}}{=} 2\int_M |\pi_B F_A|^2 \mathrm{dV}_g - 2\int_M |\pi_C F_A|^2 \mathrm{dV}_g \tag{1.54}$$

depends only on the topology of P and the $\mathfrak{Re}(\Omega)$. Explicitly, we have that for any $A \in \mathcal{A}(P)$:

$$Q(A) = \mathcal{Q}(P, \mathfrak{Re}(\Omega)) \tag{1.55}$$

where

$$\mathcal{Q}(P, \mathfrak{Re}(\Omega)) \overset{\text{def}}{=} 4\pi^2 \int_M \left[2c_2(P) - c_1^2(P)\right] \wedge \left[\mathfrak{Re}(\Omega)\right] \tag{1.56}$$

If $A_{Spin(7)}$ is a Spin(7) instanton, we obtain that:

$$\int_M |\pi_B F_{A_{Spin(7)}}|^2 \mathrm{dV}_g = \frac{1}{2}\mathcal{Q}(P, \mathfrak{Re}(\Omega)) \tag{1.57}$$

In particular, the left hand side is the same for all Spin(7) instantons. Since we are assuming that an HYM connection A_{HYM} exists, and A_{HYM} is also a Spin(7) instanton, we have that any Spin(7) instanton $A_{Spin(7)}$ satisfies:

$$\int_M |\pi_B F_{A_{Spin(7)}}|^2 \mathrm{dV}_g = \int_M |\pi_B F_{A_{HYM}}|^2 \mathrm{dV}_g = 0 \tag{1.58}$$

where the last equality follows since, by definition $\pi_B F_{A_{HYM}} = 0$. We conclude that:

$$\pi_B F_{A_{Spin(7)}} = 0. \tag{1.59}$$

Hence, any Spin(7) instanton lies in $\Lambda_0^{1,1}$ and is thus HYM. $\square$

2 Invariant Objects on Homogeneous Spaces

Let G be a Lie group. A *homogeneous space* for G is a smooth manifold equipped with a transitive G-action. The manifold $T^\star S^4$ admits a natural SO(5) action. The orbits are homogeneous spaces of codimension 1. Our strategy is to search for SO(5)-invariant instantons. This will reduce the number of variables to 1, making the problem tractable. In this section we outline the basic theory required to implement this idea.

We begin by presenting the bare minimum of general homogeneous space theory required for our purposes. We explain how to view homogeneous spaces as coset manifolds, we introduce the notion of a *naturally reductive* space, we discuss how to obtain the *canonical invariant connection* and finally we prove that the isotropy representation corresponds to the action of H on the reductive complement. We then outline the correspondence between invariant tensor fields and horizontal tensor fields over the symmetry group. Having completed our brief introduction to homogeneous spaces, we move on to homogeneous principal bundles and invariant connections. We prove their classification in detail and discuss Wang's theorem (Wang [14]). Finally, we put everything together to describe how we will handle invariant connections and their curvature forms in the following sections.

The underlying principle is that the high degree of symmetry enjoyed by a homogeeous space typically reduces differential geometric questions to representation theory.

2.1 Naturally Reductive Homogeneous Spaces and the Canonical Invariant Connection

Let G be a Lie group acting on the left of a smooth manifold M by diffeomorphisms. Suppose that the action is transitive. M is then termed a *homogeneous space* for G. Choose a reference point $p \in M$. Since M is Hausdorff, the isotropy subgroup:

$$H \stackrel{\text{def}}{=} \mathrm{Stab}_G(p) = \{g \in G \text{ such that } gp = p\} \tag{2.1}$$

is closed in G. The natural right action of H on G is smooth, free and proper. It is then an application of the quotient manifold theorem (Lee [4] p.545) that the space of left cosets:

$$G/H = \{gH \text{ such that } g \in G\} \tag{2.2}$$

inherits a unique smooth structure such that the natural projection map is a smooth submersion (Kobayashi, Nomizu [3] p.43). Choosing $p \in M$, we may diffeomorphically identify M with G/H by:

$$\begin{aligned} \phi : G/H &\xrightarrow{\sim} M \\ gH &\mapsto gp \end{aligned} \tag{2.3}$$

Thus, without loss of generality, when discussing homogeneous spaces we can restrict our attention to left coset manifolds.

Over a homogeneous space G/H there is a natural H-principal bundle. The total space is given by G and the projection map is the canonical projection to the coset space.

For our purposes, it is sufficient to consider *naturally reductive* homogeneous spaces:

Definition 2.1. A homogeneous space G/H is naturally reductive if the Lie algebra $\mathfrak{h}$ admits a vector space complement in $\mathfrak{g}$ that is stable under the restriction of Ad_G to H. The space $\mathfrak{m}$ is known as a reductive complement.

The choice of reductive complement on a naturally reductive homogeneous space endows the canonical H-bundle with a natural connection: the *canonical invariant connection*. In particular, one notices that the left invariant extension $\mathfrak{m}$ over G is Ad_H equivariant for the right H action. Furthermore, it gives a vector space complement for the left translate of $\mathfrak{h}$ over every point i.e. it is complementary to the distribution of vertical vectors. Consequently, it is a connection.

The isotropy group H stabilises the reference point p. Differentiating the action gives us a linear representation of H on T_pM. This is known as the *isotropy representation.*

The canonical invariant connection sets up a correspondence between tangent vectors on the base and tangent vectors on G. This allows us to capture the isotropy representation purely at the level of the group. We denote the representation of H on $\mathfrak{m}$ through Ad_G as $(\mathfrak{m}, \mathrm{Ad}_H)$.

Proposition 2.2. *Let G/H be a naturally reductive homogeneous space. Let $\mathfrak{m}$ be a reductive complement. The isotropy representation is isomorphic to $(\mathfrak{m}, Ad_H)$. In other words, the following square commutes for all $h \in H$:*

$$\begin{tikzcd} \mathfrak{m} \arrow[r, "\mathrm{Ad}_h"] \arrow[d, "dp_G"'] & \mathfrak{m} \arrow[d, "dp_G"] \\ T_pM \arrow[r, "dl_h"'] & T_pM \end{tikzcd}$$

Proof. The map p_G is clearly a vector space isomorphism. We only need to prove H-equivariance. We have:

$$p_G \circ l_h \circ r_{h^{-1}} = l_h \circ p_G \circ r_{h^{-1}} \tag{2.4}$$

Differentiating at the identity and applying the chain rule we obtain:

$$dp_G \circ \mathrm{Ad}_h = dl_h \circ dp_G \circ dr_{h^{-1}} \tag{2.5}$$

Finally observe that:

$$p_G \circ r_{h^{-1}} = p_G \tag{2.6}$$

and consequently:

$$dp_G \circ dr_{h^{-1}} = dp_G \tag{2.7}$$

Combining (2.5) and (2.7) completes the proof. □

2.2 Invariant Tensor Fields

We wish to describe G-invariant tensor fields over M by tensor fields over G. We select a reference point $p \in M$. Since the action is not free, the left invariant extension of a tensor T over p is not -in general- well defined. However we have an easy characterization for when it is. Essentially, the only problem is that a particular point may be connected to p through multiple group elements, each of which translate T differently. However, all of these elements lie in the same orbit of the H-action. If H acts trivially, the issue is resolved. Recalling the correspondence between the isotropy representation and the restriction of the adjoint representation to H we obtain:

Theorem 2.3. *Let G/H be a naturally reductive homogeneous space. Let $\mathfrak{m}$ be the reductive complement. G-invariant tensor fields over M correspond to elements of (appropriate tensor powers of) $\mathfrak{m}$ stabilised by Ad_H.*

Proof. We give the proof for vector fields. The proof for covariant tensors and tensors of higher rank works in exactly the same way.

Consider an invariant vector field over the base and let X be its value at the identity coset. Let $v \in \mathfrak{m}$ be the unique horizontal vector projecting to X. Since X is invariant, it is stabilised by the isotropy representation. Consequently v is stabilised by the adjoint action of H.

Conversely, suppose that $v \in \mathfrak{m}$ is stabilised by Ad_h. Let X be its projection to the base. X is then stabilised by the isotropy representation and its left invariant extension is well defined. In particular we set:

$$X_{|_{gH}} \stackrel{\text{def}}{=} dl_g X \tag{2.8}$$

It is clear that the two constructions are inverse to each other. □

2.3 Homogeneous Principal Bundles

Let $M = G/H$ be a homogeneous space. Let S be a Lie group. We are interested in studying principal S-bundles over M that are compatible with its symmetry. We begin with the following definition:

Definition 2.4. A homogeneous S-bundle over M is a principal S-bundle equipped with a left action $G \mapsto \mathrm{Aut}(P)$ that lifts the action on M. Explicitly, for any $g \in G$ we have the following commutative square:

$$\begin{tikzcd} P \arrow[r, "l_g"] \arrow[d, "\pi"'] & P \arrow[d, "\pi"] \\ M \arrow[r, "l_g"'] & M \end{tikzcd}$$

The following observation is immediate from the definition.

Proposition 2.5. *Let G and S be Lie groups. Let M be a homogeneous space for G. Let P be a homogeneous principal S-bundle over M. Then P is a homogeneous space for $G \times S$.*

Proof. We need to display a transitive left action of $G \times S$ on P. Define:

$$(g, s)p \overset{\text{def}}{=} gps^{-1} \tag{2.9}$$

This is clearly transitive. Since the left G-action lifts the action on M, it is fiber-transitive. Since P is a principal S-bundle, the right S action is transitive on each fiber. ☐

We now compute the stabiliser of this action. We have:

$$gps^{-1} = p \iff gp = ps \tag{2.10}$$

Since the action of S is fiber preserving and the action of G lifts the action on M, we must have that:

$$g = h \in H = \mathrm{Stab}_G(\pi(p)). \tag{2.11}$$

Now, since elements of H preserve the fiber pS, for each $h \in H$ there is a unique $s \in S$ such that:

$$hp = ps \tag{2.12}$$

This gives us a Lie group homomorphism $H \to S$ uniquely defined by the equation:

$$hp = p\lambda(h) \tag{2.13}$$

This is frequently referred to as the *isotropy homomorphism* and as we shall see below its element-conjugacy class determines P. We now observe that:

$$\mathrm{Stab}_{G\times S}(p) = \big\{(h, \lambda(h)) \text{ such that } h \in H\big\} < G \times S \tag{2.14}$$

It follows that (at the level of smooth manifolds):

$$P \cong \frac{G \times S}{H} \tag{2.15}$$

Using λ we may define the following left action of H on S:

$$hs \overset{\text{def}}{=} s\lambda(h)^{-1} \tag{2.16}$$

Using the structure of G as an H-bundle over M, the action (2.16) allows us to define the associated fiber bundle with standard fiber S:

$$G \times_{(H,\lambda)} S = (G \times S)/\sim\ , \text{ where } (g, s) \sim (gh, h^{-1}s) \tag{2.17}$$

With this definition, we have that:

$$P \cong \frac{G \times S}{H} = G \times_{(H,\lambda)} S \tag{2.18}$$

from which we see that the associated fiber bundle inherits the structure of an S-principal bundle.

We now draw from this construction to classify homogeneous bundles up to their natural notion of isomorphism. This is as follows:

Definition 2.6. A homogeneous bundle isomorphism is a G-equivariant principal bundle isomorphism.

The following theorem classifies homogeneous bundles up to homogeneous bundle isomorphism. Note that this is a finer classification than the usual one (i.e. multiple isomorphism classes of homogeneous bundles might lie in the same principal bundle isomorphism class).

Theorem 2.7. *Let G and S be Lie groups. Let M be a homogeneous space for G. Let H be the isotropy group (we don't care about a particular inclusion of H in G so we do not have to specify a reference point). Homogeneous principal S-bundles over M are classified by element-conjugacy classes of Lie group homomorphisms:*

$$\lambda : H \to S$$

In view of this classification we denote the homogeneous bundle corresponding to λ by P_λ.

Proof. A homogeneous bundle P can be diffeomorphically identified with $G \times_{(H,\lambda)} S$ as in (2.18). The latter may be promoted to an S-principal bundle by transporting the necessary structure from P. Unwinding the identifications we see that the diffeomorphism is given by:

$$\phi([g,s]) = gps^{-1} \tag{2.19}$$

We use ϕ to transport the structure of P to $G \times_{(H,\lambda)} S$. In particular, we define:

$$[g,s]s' \overset{\text{def}}{=} [g, s'^{-1}s] \tag{2.20}$$

$$g'[g,s] \overset{\text{def}}{=} [g'g, s] \tag{2.21}$$

With these definitions, ϕ becomes an isomorphism of homogeneous principal S-bundles. Indeed:

$$\phi([g,s]s') = \phi([g,s'^{-1}s]) = gp(s'^{-1}s)^{-1} = gps^{-1}s' = \phi([g,s])s' \tag{2.22}$$

$$\phi(g'[g,s]) = \phi([g'g,s]) = g'gps^{-1} = g'(\phi([g,s])) \tag{2.23}$$

Conversely, given $\lambda : H \to S$, we obtain a left H-action on S by:

$$hs \overset{\text{def}}{=} s\lambda(h)^{-1} \tag{2.24}$$

We then set

$$P_\lambda \overset{\text{def}}{=} G \times_{(H,\lambda)} S \tag{2.25}$$

This is a smooth fiber bundle with fiber S over M. We upgrade it to a homogeneous principal bundle by declaring:

$$[g,s]s' \overset{\text{def}}{=} [g, s'^{-1}s] \tag{2.26}$$

$$g'[g,s] \overset{\text{def}}{=} [g'g, s] \tag{2.27}$$

It is clear that (2.27) and (2.26) are well defined and that (2.26) lifts the action of G on M. One can check that (2.27) is compatible with the trivializations of P_λ coming from its structure as a fiber bundle.

We have established that any Lie group homomorphism $\lambda : H \to S$ gives rise to a homogeneous principal S-bundle P_λ. Furthermore, we have seen that all such bundles can be put into this form.

In fact, we have given a procedure to achieve that. It is natural to ask whether this procedure will return λ when applied to P_λ. If we use $[1, 1]$ as our reference point we indeed retrieve λ. In other words it holds that:

$$h[1, 1] = [1, 1]\lambda(h) \tag{2.28}$$

If we pick another reference point in the same fiber, we obtain a different isotropy homomorphism $\mu \neq \lambda$. Nevertheless, we notice that μ and λ are always related by conjugation with a fixed element of s. Motivated by this observation, we claim that:

$$G \times_{(H,\lambda)} S \cong G \times_{(H,\mu)} S \iff \mu \text{ and } \lambda \text{ are element-conjugate.} \tag{2.29}$$

We first prove the rightward implication. The proof will suggest how to treat the converse.

Suppose that:

$$\phi : P_\lambda \xrightarrow{\sim} P_\mu \tag{2.30}$$

is a homogeneous principal bundle isomorphism. Since bundle isomorphisms are fiber preserving, there is an $s \in S$ such that:

$$\phi([1, 1]_\lambda) = [1, s]_\mu = [1, 1]_\mu s^{-1} \tag{2.31}$$

Recall that we have the equations:

$$h[1, 1]_\lambda = [1, 1]_\lambda \lambda(h) \tag{2.32}$$

$$h[1, 1]_\mu = [1, 1]_\mu \mu(h) \tag{2.33}$$

Apply ϕ to the first equation and use (2.31) to obtain:

$$h[1, 1]_\mu s^{-1} = [1, 1]_\mu s^{-1}\lambda(h) \tag{2.34}$$

Note that we have used that ϕ is a G-equivariant bundle morphism. Now use the second equation to cross h over on the left hand side. This gives:

$$[1, 1]_\mu \mu(h)s^{-1} = [1, 1]_\mu s^{-1}\lambda(h) \tag{2.35}$$

Since the right S action on any principal S bundle is free, we obtain:

$$\mu(h)s^{-1} = s^{-1}\lambda(h) \tag{2.36}$$

which we can write as:

$$\mu(h) = s^{-1}\lambda(h)s \tag{2.37}$$

For the converse, suppose that 2.37 holds. Define:

$$\phi([1, 1]_\lambda) = [1, s]_\mu \tag{2.38}$$

and extend the map to all of P_λ by the $G \times S$ action. By virtue of 2.37, this extension is well defined. It is clear that it is a G-equivariant bundle automorphism. □

Having classified homogeneous bundles, we collect the various agents involved in their study in one useful diagram. Let P be a homogeneous S bundle. Let $\pi : P \twoheadrightarrow M$ denote the projection map. Since P is homogeneous, we have compatible left G actions on M and P. Choosing a reference point $x \in M$ and a reference point $p \in P$ over x, we obtain maps:

$$p_G : G \to M$$
$$\Phi : G \to P$$

The compatibility of the actions gives:

$$p_G = \pi \circ \Phi \tag{2.39}$$

Recall that P is a homogeneous space for $G \times S$. The choice of p as a reference point on P provides us with a map:

$$p_{G\times S} : G \times S \to P$$

By the definition of the action of $G \times S$ on P we immediately see that:

$$\Phi = p_{G\times S} \circ \iota \tag{2.40}$$

where ι denotes the inclusion in the first factor.

Using the chosen reference points, we can recast this description in terms of coset manifolds. This way, whenever we work with a homogeneous S bundle over a homogeneous space G/H, we have the following commutative diagram:

$$\begin{tikzcd}
 & G \times S \arrow[d, two heads, "p_{G\times S}"] \\
G \arrow[ur, hook, "\iota"] \arrow[r, "\Phi"'] \arrow[d, two heads, "p_G"'] & G \times_{(H,\lambda)} S \arrow[dl, two heads, "\pi"] \\
G/H &
\end{tikzcd}$$

2.4 Vector Valued ρ-Invariant Forms

Our ultimate goal is to study invariant connections and invariant curvature forms on homogeneous principal bundles. We begin with a definition:

Definition 2.8. Let P_λ be a homogeneous S-bundle over a homogeneous space G/H. A tensorial k-form of type Ad is invariant if it is G-invariant as a k-form on P_λ. A connection $A \in \mathcal{A}(P_\lambda)$ is invariant if it is G-invariant as a 1-form on P_λ.

Tensorial forms and connections -regardless of whether or not they are invariant- satisfy a right equivariance property with respect to the right S-action on the bundle. In particular, we have:

$$r_s^\star \omega = \mathrm{Ad}_{s^{-1}}(\omega) \tag{2.41}$$

We may capture property (2.41) and G-invariance simultaneously by introducing a representation ρ of $G \times S$ on $\mathfrak{s}$ such that invariant tensorial forms correspond to *ρ-invariant forms.*

In what follows, we work for a general representation of some group G on a vector space V. When we return to the setting of homogeneous bundles, the role of G will be played by $G \times S$ and the role of V will be played b $\mathfrak{s}$.

Definition 2.9. Let $M = G/H$ be a homogeneous space. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. A form $\omega \in C^\infty\left(\Lambda^k T^\star M \otimes V\right)$ is ρ-invariant if:

$$l_g^\star \omega = \rho_g(\omega) \tag{2.42}$$

We can capture ρ-invariant forms as forms over G satisfying a certain condition at the identity.

Proposition 2.10. *Let $M = G/H$ be a naturally reductive homogeneous space. Let $\mathfrak{m}$ be the reductive complement. Let $\rho : G \to GL(V)$ be a representation. The ρ-invariant V-valued forms on M correspond to elements $\alpha \in \Lambda^k \mathfrak{m}^\star \otimes V$ satisfying:*

$$Ad_g^\star \omega = \rho_g(\omega) \tag{2.43}$$

Proof. Given α satisfying (2.43) we can extend it over G by declaring:

$$\alpha_{|_g} = \rho_{g^{-1}}\left(l_{g^{-1}}^\star \alpha\right) \tag{2.44}$$

Condition (2.43) guarantees that this extension is well defined. Furthermore, one easily observes that it is right H-invariant and hence descends to the base:

$$r_h^\star \alpha = \alpha \text{ for all } h \in H$$

Given a ρ invariant form over the base, we can pull it back to G through the projection map $G \mapsto G$. The pullback satisfies (2.43) at the identity. $\square$

2.5 Invariant Connections and Wang's Theorem

We will classify connections on homogeneous principal S-bundles that are invariant under the action of G. The following theorem is due to Wang (Wang [14] p.8).

Theorem 2.11. *Let $M = G/H$ be a naturally reductive homogeneous space. Let $\mathfrak{m}$ be a reductive complement. Let P_λ be a homogeneous S-bundle over M. There is a one to one correspondence between invariant connections A over P_λ and linear maps:*

$$\Lambda : \mathfrak{m} \to \mathfrak{s} \tag{2.45}$$

satisfying:

$$\Lambda \circ Ad_h = Ad_{\lambda(h)} \circ \Lambda \text{ for any } h \in H \tag{2.46}$$

Proof. Given Λ as in the proposition, one extends it to:

$$\Lambda + d\lambda_{|_1} : \mathfrak{g} \to \mathfrak{s}. \tag{2.47}$$

One then defines:

$$\Omega : T_{(1,1)}(G \times S) = \mathfrak{g} \oplus \mathfrak{s} \to \mathfrak{s} \tag{2.48}$$

$$\Omega = \mathrm{Id}_{\mathfrak{s}} - \Lambda - d\lambda_{|_1} \tag{2.49}$$

We define a representation of $G \times S$ on $\mathfrak{s}$ by:

$$\rho : (g, s) \mapsto \mathrm{Ad}_s \tag{2.50}$$

The representation ρ has been set up so that the ρ invariant forms are both right S Ad-equivariant and left G-invariant.

The ρ-invariant forms on P_λ then correspond to forms over $T_e(G \times S)$ satisfying 2.43 and sending the Lie algebra of the isotropy subgroup to 0.

The element Ω satisfies the requisite conditions. By proposition , its ρ-invariant extension over $G \times S$ is then a pullback of a form over P_λ. This form maps vertical vectors to the identity, is right S-equivariant and left G-invariant. It is therefore an invariant connection on P_λ.

For the converse, we simply pull the connection form back to G (through the map Φ from the diagram at the end of section 2.3), we restrict the result to the identity and finally we subtract the connection form of the canonical invariant connection. □

Note that in the above classification, the canonical invariant connection corresponds to $\Lambda = 0$.

2.6 Tensorial Forms at the Level of the Symmetry Group

The last ingredient we require in order to handle invariant connections effectively is to understand how to work with $\mathrm{ad}(P_\lambda)$ forms over the base at the level of the symmetry group G. We will use the notation of the commutative diagram at the end of section 2.3.

We know (Tu [12] p.278) that $\mathrm{ad}(P_\lambda)$ valued forms over M correspond to tensorial forms of type Ad over P_λ. This correspondence is simple to describe. The pullback of $\mathrm{ad}(P_\lambda)$ on P_λ is trivial and to an $\mathrm{ad}(P_\lambda)$-valued form β we associate its pullback $\pi^\star\beta$.

In fact, the map Φ allows us to set up a third agent in this correspondence. The pullback of $\mathrm{ad}(P_\lambda)$ over G is trivial:

$$p_G^\star \mathrm{ad}(P_\lambda) \cong G \times \mathfrak{s} \tag{2.51}$$

A trivialization is given by identifying the fiber over $g \in G$ with $\mathfrak{s}$ as follows:

$$\mathrm{ad}(P_\lambda)_{|_{p_G(g)}} \xrightarrow{\sim} \mathfrak{s} \tag{2.52}$$

$$u \mapsto v \text{ such that } u = [\Phi(g), v] \tag{2.53}$$

The pullback to G of $\beta \in C^\infty(\Lambda^k T^\star M \otimes \mathrm{ad}(P_\lambda))$ is a form with values in $p_G^\star \mathrm{ad}(P_\lambda)$. Using the identification (2.53), we view it as a form with values in $\mathfrak{s}$. It can be easily checked that the form obtained through this process agrees with $\Phi^\star \pi^\star \beta$.

We conclude that there is a 3-part correspondence among tensorial $\mathfrak{s}$-valued forms over P_λ, $\mathfrak{s}$-valued horizontal forms (with respect to the canonical connection) over G satisfying:

$$r_h^\star \gamma = \mathrm{Ad}_{h^{-1}}(\gamma) \tag{2.54}$$

and $\mathrm{ad}(P_\lambda)$-valued forms over M.

Among these forms, the invariant ones have a simple characterization: when written on G they are left invariant. Consequently, they are determined by their value at the identity. This is essentially a restatement of Wang's theorem. In fact, the condition in Wang's theorem is precisely the condition required for the left invariant extension of $\beta \in \Lambda^k \mathfrak{m}^\star \otimes \mathfrak{s}$ to satisfy (2.54).

2.7 Working with Invariant Connections on Homogeneous Bundles

Having classified homogeneous bundles and their invariant connections, we collect the ideas of the preceding sections in a compact account of how we will treat these objects in the following chapters.

We will refer to the commutative diagram at the end of section 2.3 and the language used in the proof of Wang's theorem.

The group $G \times S$ acts on $\mathfrak{s}$ by:

$$\rho : (g, s) \mapsto \mathrm{Ad}_s \tag{2.55}$$

An invariant connection A is a ρ-invariant $\mathfrak{s}$-valued form over P_λ. The corresponding element of $(\mathfrak{m} \oplus \mathfrak{s})^\star$ promised by proposition 2.5 can be written as:

$$\Omega \stackrel{\mathrm{def}}{=} \left(p_{G\times S}^\star A\right)_{|(1,1)} = \mathrm{Id}_s - d\lambda_{|_e} - \Lambda \tag{2.56}$$

We will always use the canonical invariant connection as a reference. As an element of $(\mathfrak{m} \oplus \mathfrak{s})^\star$ we have:

$$\Omega_{\mathrm{ref}} \stackrel{\mathrm{def}}{=} \left(p_{G\times S}^\star A_{\mathrm{ref}}\right)_{|(1,1)} = \mathrm{Id}_s - d\lambda_{|_e} \tag{2.57}$$

We then write:

$$A = A_{\mathrm{ref}} + \alpha. \tag{2.58}$$

We define:

$$\omega \stackrel{\mathrm{def}}{=} \left(\pi_{G\times S}^\star \alpha\right)_{|(1,1)} = -\Lambda$$

so that:

$$\Omega = \Omega_{ref} + \omega. \tag{2.59}$$

We also denote by $\Omega, \Omega_{\mathrm{ref}}$ and ω the ρ-invariant extensions of these elements over $G \times S$. These are the pullbacks of A, A_{ref} and α respectively.

Furthermore, we can consider the left invariant extension over G of the following maps:

$$-d\lambda - \Lambda : \mathfrak{g} \to \mathfrak{s} \tag{2.60}$$

$$-\Lambda : \mathfrak{g} \to \mathfrak{s} \tag{2.61}$$

Denote the resulting forms as Ω_G and ω_G respectively. It is trivial to check that:

$$\Omega_G = \iota^\star p_{G\times S}^\star A \tag{2.62}$$

$$\omega_G = \iota^\star p_{G\times S}^\star \alpha \tag{2.63}$$

From the commutativity of the diagram at the end of section 2.3, we obtain:

$$\Omega_G = \Phi^\star A \tag{2.64}$$

$$\omega_G = \Phi^\star \alpha \tag{2.65}$$

Te curvature of A is given by:

$$F_A = dA + \frac{1}{2}[A \wedge A] \tag{2.66}$$

The operations involved behave well under pullback and we obtain:

$$\Phi^\star F_A = d\Omega_G + \frac{1}{2}[\Omega_G \wedge \Omega_G] \tag{2.67}$$

We always describe the invariant connection A by the map $-\Lambda$. In other words, we choose to work over G in the 3-part correspondence outlined in section 2.6. The map $-\Lambda$ corresponds to the tensorial form α written over G as ω_G. The full connection is then given by Ω_G. It is also a left invariant form satisfying the transformation law (2.54). It does not correspond to a tensorial form on P_λ. This is reflected by the fact that it is not horizontal. Its value at the identity is:

$$\Omega_{G_{|1}} = -d\lambda - \Lambda. \tag{2.68}$$

In examples it is typically easy to compute $-d\lambda - \Lambda$ in terms of some bases of the relevant Lie algebras. The curvature F_A is tensorial and as such it can be viewed as a form over the base, a form over P_λ, or a form over G. We choose to work over G. We then have:

$$F_A = d\Omega_G + \frac{1}{2}[\Omega_G \wedge \Omega_G] \tag{2.69}$$

The first term is easily computed using the Mauer-Cartan relations so long as we know the structure constants for the bracket of $\mathfrak{g}$. The second term is easily computed provided we know the structure constants for the bracket of $\mathfrak{s}$.

3 The Manifold $T^{\star}S^4$

In this section we introduce the natural cohomogeneity-one S0(5)-action on $T^{\star}S^4$ and derive general formulae expressing SO(5)-invariant Kähler structures coming from a global SO(5)-invariant Kähler potential in terms of invariant forms. The overall technique for finding invariant objects in cohomogeneity one is essentially the same as in the paper (Lotay-Oliveira [7]). The calculations for SO(4)-invariant Kähler structures on $T^{\star}S^3$ have been carried out in the paper (Oliveira [8]). Our notation is the same as the one employed there.

3.1 Models for $T^{\star}S^4$ and the Cohomogeneity One SO(5) Action

We will work with two different models for the space $T^{\star}S^4$. It is crucial to introduce both of them as they capture different aspects of the structures we wish to study. The first model elucidates the SO(5) symmetry, whereas the second demonstrates the complex structure.

3.1.1 The Real Model and the SO(5) Action

The manifold $T^{\star}S^4$ admits a natural embedding into $\mathbb{R}^{10}$ as follows:

$$T^{\star}S^4 = \left\{(x, y^{\intercal}) \ \mid \ \langle x, y\rangle = 0\right\} \subset T^{\star}\mathbb{R}^4$$

where we naturally split points of $\mathbb{R}^{10} = T^{\star}\mathbb{R}^5$ in two parts, the second of which is a row vector -demonstrating that it is a covector-.

There is a natural left action of SO(5) defined as:

$$g\,(x, y^{\intercal}) \stackrel{\text{def}}{=} \left(gx, y^{\intercal}g^{-1}\right) \tag{3.1}$$

This corresponds to the natural action of SO(5) on S^4 with the induced pullback operation on the covector part.

We now compute the orbits and isotropy groups for the action 3.1.

Proposition 3.1. *The principal orbits of 3.1 are the positive radius sphere bundles in $T^{\star}S^4$. They have isotropy group isomorphic to $SO(3)$. The singular orbit is S^4 sitting in its cotangent bundle as the zero section. Its isotropy subgroup is isomorphic to SO(4).*

Proof. Let $p_{R_-} = \left(x, y_{R_-}\right) \in T^{\star}S^4$ be the point:

$$x \stackrel{\text{def}}{=} (1,0,0,0,0)^{\intercal}\,, \quad y_{R_-} \stackrel{\text{def}}{=} (0, R_-, 0, 0, 0)\,. \tag{3.2}$$

An element $g \in SO(5)$ stabilizes p_{R_-} if and only if:

$$gx = x \text{ and } yg^{-1} = y$$

The first equation forces the first column of g to vanish. Since the columns are orthonormal, this forces g to lie in the lower right diagonal copy of SO(4). Repeating the same argument, we see that the second equation forces the SO(4) block to lie in the lower right diagonal copy of $SO(3)$. Matrices lying in this subgroup definitely stabilise p_{R_-} and hence we have:

$$\mathrm{Stab}_{SO(5)}(p_{R_-}) = SO(3) \tag{3.3}$$

Since the action of SO(5) is transitive on S^4 and the action of SO(4) is transitive on S^3, the orbit of p_{R_-} is precisely the R_--sphere bundle in $T^{\star}S^4$:

$$O_{R_-} = SO(5)p_{R_-} = S_{R_-}\left(T^{\star}S^4\right) = \frac{SO(5)}{SO(3)} \tag{3.4}$$

Now work with the point:

$$p_0 = (x, 0)$$

where x is as in 3.2. Applying the same argument used for the positive radius case we see that:

$$\mathrm{Stab}_{SO(5)}(p_0) = SO(4) \tag{3.5}$$

Using the same reasoning as above, we immediately see that the orbit is the zero section:

$$SO(5)p_0 = S^4 = \frac{SO(5)}{SO(4)} \tag{3.6}$$

□

For any vector bundle E over a manifold M, we can write the following decomposition at the topological level:

$$E - M \cong (0, \infty) \times S(E)$$

where $S(E)$ denotes the unit sphere bundle of E. In our case, this splitting takes the form:

$$T^\star S^4 - S_4 \cong (0, \infty) \times S_1 \left(T^\star S^4\right) \tag{3.7}$$

where the subscript 1 denotes the radius of the sphere in the ambient Euclidean metric of $\mathbb{R}^{10}$. This identification is explicitly given by:

$$(R_-, \omega) \mapsto R_-\omega \tag{3.8}$$

where $\omega \in S_1\left(T^\star S^4\right)$ and $R_- > 0$.

Equation 3.2 provides us with a natural choice of reference point on each principal orbit O_{R_-}. This choice identifies O_{R_-} with the left coset space $\frac{SO(5)}{SO(3)}$ as in 3.4. Combining the above we may write:

$$T^\star S^4 - S_4 \cong (0, \infty) \times \frac{SO(5)}{SO(3)}$$

where:

$$\left(R_-,\ gSO(5)\right) \mapsto gp_{R_-} \tag{3.9}$$

Note however, that the unit sphere bundle is twisted as can be shown, for instance, by the hairyball theorem.

3.1.2 The Complex Quadric Model

We may also realise $T^\star S^4$ as a complex submanifold of $\mathbb{C}^5$. Consider the quadratic polynomial:

$$F \stackrel{\text{def}}{=} z_1^2 + \ldots + z_5^2 \tag{3.10}$$

Since F is holomorphic, we may compute its derivative as:

$$dF = \partial F = \sum_{j=1}^{5} 2z_i dz^i \tag{3.11}$$

Since every point p in $F^{-1}(1)$ must have a non-zero coordinate, $dF_{|p}$ does not vanish. It follows that 1 is a regular value for F. Since F is holomorphic, we see that:

$$X^8 \stackrel{\text{def}}{=} F^{-1}(1) \tag{3.12}$$

is a complex submanifold of $\mathbb{C}^5$ and hence Kähler.

We split the complex coordinates of $\mathbb{C}^5$ into their real and imaginary parts:

$$z_j = x_j + iy_j.$$

We introduce the functions:

$$r^2 \stackrel{\text{def}}{=} |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2. \tag{3.13}$$

$$R_+^2 \stackrel{\text{def}}{=} x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2. \tag{3.14}$$

$$R_-^2 \stackrel{\text{def}}{=} y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2. \tag{3.15}$$

The following relations follow:

$$R_+ = \frac{r^2+1}{2}, \quad R_- = \frac{r^2-1}{2}, \quad r^2 = R_+^2 + R_-^2. \tag{3.16}$$

Define the map:

$$\begin{aligned} \Psi : \mathbb{C}^5 &\to \mathbb{R}^{10} \\ (z_1, ..., z_5) &\mapsto \left(\frac{x}{R_+}, y^\intercal \right) \end{aligned} \tag{3.17}$$

It may be easily seen that this cuts down to a diffeomorphism:

$$X^8 \xrightarrow{\sim} T^\star S^4$$

We therefore conclude that:

Proposition 3.2. *The complex quadric X^8 is diffeomorphic to the total space $T^\star S^4$.*

This identification endows $T^\star S^4$ with a natural complex structure. Viewing the latter as a complex quadric hypersurface is more suited to complex geometric calculations. Seeing as we are interesting in studying $T^\star S^4$ as a CY 4-fold, from here on we mostly work in the complex model.

The minimum value of r on X^8 is $r = 1$ and the associated level set corresponds to the singular orbit $\frac{SO(5)}{SO(4)}$. The latter sits inside X^8 as an embedded totally real submanifold (Patrizio [9]). When working in the complex model, we will modify the notation of the previous subsection and relabel the point p_{R_-} by p_r. We then have:

$$p_r \stackrel{\text{def}}{=} (R_+, iR_-, 0, 0, 0)^\intercal \tag{3.18}$$

With this definition p_r corresponds to p_{R_-} under the identification 3.17. Furthermore, when working in the complex model, we will denote the principal orbit at radius $r > 1$ as O_r rather than O_{R_-}.

For the sake of completeness, we mention that it is customary in the literature to introduce the variable $t \in [0, \infty)$ implicitly defined by the relation:

$$\cosh(t) \stackrel{\text{def}}{=} r^2 \tag{3.19}$$

The point is that the function $t \circ r^2$ is then a plurisubharmonic exhaustion of X^8, constant on the principal orbits, equal to 1 on the singular orbit and satisfying the Monge–Ampère equation:

$$\left(\partial\overline{\partial} t(r^2) \right)^n = 0. \tag{3.20}$$

Furthermore, these three properties uniquely determine $t \circ r^2$ up to scaling (Patrizio, [9]).

The following proposition is an immediate consequence of 3.11.

Proposition 3.3. *At a point $p \in X^8 \subset \mathbb{C}^5$, with $p_5 \neq 0$ we have:*

$$T_pX^8 = \left\{ \left(v_1, v_2 v_3, v_4, -\frac{1}{p_5} (p_1 v_1 + p_2 v_2 + p_3 v_3 + p_4 v_4) \right)^\intercal \ s.t.\ v_j \in \mathbb{C} \right\}. \tag{3.21}$$

Our final task in this subsection is to introduce a natural choice of a radial vector field ∂_r, compatible with the function r and the splitting 3.7.

Proposition 3.4. *There exists a unique smooth vector field ∂_r on $X^8 - S^4$ characterised by the following properties:*

1. *The vector field ∂_r is tangent to $(0,\infty)$ in the splitting 3.7.*
2. *$dr\left(\partial_r\right) = 1$.*

Let $(x,y) \in X^8$. The vector field ∂_r can be expressed as follows in terms of the standard coordinate vector fields on $\mathbb{C}^5$:

$$\partial_{r_{|(x,y)}} = \frac{r}{2R_+^2}\left(\sum_{j=1}^{5} x^j \partial_{x^j_{|(x,y)}}\right) + \frac{r}{2R_-^2}\left(\sum_{j=1}^{5} y^j \partial_{y^j_{|(x,y)}}\right) \tag{3.22}$$

Proof. It serves intuition to begin working in the real model. It is obvious that over the point $p = (\tilde{x}, \tilde{y}) \in T^\star S^4 \subset \mathbb{R}^{10}$, we have:

$$T_p(0,\infty) = \mathrm{Span}_{\mathbb{R}}\left((0,\tilde{y})\right)$$

We translate this to the complex model. The inverse of the identification 3.17 is given by:

$$\Psi^{-1} : (\tilde{x}, \tilde{y}^\intercal) \mapsto \left(R_+(\tilde{y})\tilde{x}, \tilde{y}\right) = \left(\left(R_-(\tilde{y})^2 + 1\right)^{\frac{1}{2}} \tilde{x}, \tilde{y}\right) = \left((|\tilde{y}|^2 + 1)^{\frac{1}{2}} \tilde{x}, \tilde{y}\right)$$

We compute the derivative of this map and apply it to the vector $(0, y)$ to obtain:

$$\left(d\Psi^{-1}_{|(\tilde{x},\tilde{y})}\right)(0,\tilde{y}) = \left(\frac{R_-^2}{R_+}\tilde{x}, \tilde{y}\right).$$

Consequently, we have:

$$T_{\Psi^{-1}(p)}(0,\infty) = \mathrm{Span}_{\mathbb{R}}\left(\frac{R_-^2}{R_+}\tilde{x}^j \partial_{x^j} + \tilde{y^j}\partial_{y^j}\right)$$

We therefore obtain the following expression for the radial tangent space at $(x,y) \in X^8$:

$$T_{(x,y)}(0,\infty) = \mathrm{Span}_{\mathbb{R}}\left(\frac{R_-^2}{R_+^2}x^j \partial_{x^j} + y^j\partial_{y^j}\right)$$

We now determine how to scale this vector so as to achieve the normalization condition (2) in the statement of the proposition. We compute:

$$\begin{aligned} dr\left(\frac{R_-^2}{R_+^2}x^j \partial_{x^j} + y^j\partial_{y^j}\right) &= \frac{R_-^2}{R_+^2}x^j dr\left(\partial_{x^j}\right) + y^j dr\left(\partial_{y^j}\right) \\ &= \frac{R_-^2}{R_+^2}\sum_{j=1}^{5}\frac{(x^j)^2}{r} + \sum_{j=1}^{5}\frac{(y^j)^2}{r} = \frac{2R_-^2}{r} \end{aligned}$$

We scale by the above quantity to achieve the desired normalization and discover that we should define ∂_r by the formula 3.22. It is then clear that this choice of ∂_r satisfies the required conditions. □

3.1.3 Adjoint Representation, Reductive Splitting and Invariant Tensors

We now describe how to work with invariant tensors on X^8 at the level of SO(5). This involves specifying the reductive splitting for the principal orbits, giving the irreducible decomposition of the adjoint representation of $SO(3)$ on the reductive complement $\mathfrak{m}$ and identifying the geometric meaning of the vectors in $\mathfrak{m}$.

The Lie algebra $\mathfrak{so}(5)$ consists of all 5×5 antisymmetric matrices under the commutator bracket. It is given by:

$$\mathfrak{so}(5) = \mathrm{Span}\left\{C_{ij} \mid 1 \le i < j \le 5\right\}$$

where $C_{ij} = e_{ij} - e_{ji}$ and e_{ij} is the matrix with ij entry equal to 1 and all other entries vanishing. The bracket is characterized by:

$$\left[C_{ij}, C_{ik}\right] = -C_{jk} \tag{3.23}$$

$$\left[C_{ij}, C_{kl}\right] = 0 \text{ for } i \ne j \ne k \ne l \tag{3.24}$$

We write:

$$X_1 = C_{12},\ X_2 = C_{13},\ X_3 = C_{14},\ X_4 = C_{15},\ X_5 = C_{23}$$
$$X_6 = C_{24},\ X_7 = C_{25},\ X_8 = C_{34},\ X_9 = C_{35},\ X_{10} = C_{45}.$$

and we denote the dual one-form of X_i by θ^i.
The adjoint representation of SO(5) on is Lie algebra can be restricted to $SO(3) < SO(5)$. An element $g \in SO(3)$ then acts on $A \in \mathfrak{so}(5)$ by conjugation. It can be easily seen that this representation splits as:

$$\mathfrak{so}(5) = \langle X_1\rangle \oplus \langle X_2, X_3, X_4\rangle \oplus \langle X_5, X_6, X_7\rangle \oplus \langle X_8, X_9, X_{10}\rangle \tag{3.25}$$

The first summand is the trivial representation and the other three summands are isomorphic to the vector representation of $SO(3)$ on $\mathbb{R}^3$ (the order in which the vectors appear corresponds to the standard basis (e_1, e_2, e_3) of $\mathbb{R}^3$).

Each principal orbit $O_r = \frac{SO(5)}{SO(3)}$ is a homogeneous space. We may thus work with invariant tensors at the level of the symmetry group as described in section 2. For this purpose we need to select a preferred point on O_r. We will use the point p_r introduced in 3.18. As explained in section 3.1, the stabiliser of p_r is the lower right copy of $SO(3)$. Consequently, its Lie algebra is given by:

$$\mathfrak{so}(3) = \langle X_8, X_9, X_{10}\rangle \tag{3.26}$$

We define the natural reductive complement:

$$\mathfrak{m} = \langle X_1\rangle \oplus \langle X_2, X_3, X_4\rangle \oplus \langle X_5, X_6, X_7\rangle \tag{3.27}$$

Owing to 3.25, it is closed under the action of $Ad_{SO(3)}$ and its left invariant extension gives the canonical invariant connection on the $SO(3)$-bundle:

$$SO(3) \hookrightarrow SO(5) \twoheadrightarrow O_r$$

Given our choice of p_r, the right arrow is given by:

$$\pi : g \mapsto gp_r \tag{3.28}$$

Using 3.1 and 3.17 we find that:

$$d\pi_{|\mathrm{Id}} : \mathfrak{m} \xrightarrow{\sim} T_{p_r}O_r$$

acts on a matrix A by:

$$A \mapsto \left(R_+ c_1(A), -R_- r_2(A)\right) \tag{3.29}$$

where $c_1(\cdot)$ denotes the operation of taking the first column and $r_2(\cdot)$ denotes the operation of taking the second row. Using 3.29 we obtain the equations:

$$d\pi_{|\mathrm{Id}} X_1 = -R_+ \partial_{x^2_{|p_r}} + R_- \partial_{y^1_{|p_r}} \tag{3.30}$$

$$d\pi_{|\mathrm{Id}} X_2 = -R_+ \partial_{x^3_{|p_r}} \tag{3.31}$$

$$d\pi_{|\mathrm{Id}} X_3 = -R_+ \partial_{x^4_{|p_r}} \tag{3.32}$$

$$d\pi_{|\mathrm{Id}} X_4 = -R_+ \partial_{x^5_{|p_r}} \tag{3.33}$$

$$d\pi_{|\mathrm{Id}} X_5 = -R_- \partial_{y^3_{|p_r}} \tag{3.34}$$

$$d\pi_{|\mathrm{Id}} X_6 = -R_- \partial_{y^4_{|p_r}} \tag{3.35}$$

$$d\pi_{|\mathrm{Id}} X_7 = -R_- \partial_{y^5_{|p_r}} \tag{3.36}$$

It is evident from these expressions that X_1, X_2, X_3, X_4 correspond to infinitesimal motions in the horizontal direction on the base S^4 and $X_5, X6, X_7$ correspond to infinitesimal vertical motions along the fiber of the sphere bundle. Of all these vectors, only X_1 is invariant under $Ad_{SO(3)}$ and thus extends to a globally defined, SO(5)-symmetric vector field over O_r (for $r > 1$).

From here on we will suppress denoting the operator $d\pi$ and the subscript indicating that we are working over p_r. Thus, over p_r, tensors can be written as linear combinations of tensor products of the $X_i^{'s}$ and the $\theta^{i's}$ irrespectively of whether they are stabilised by $Ad_{SO(3)}$.

Evaluating the expression 3.22 at p_r we obtain:

$$\partial_r = \frac{r}{2R_+}\partial_{x^1} + \frac{r}{2R_-}\partial_{y^2} \tag{3.37}$$

Using 3.30-3.37 we conclude that (at p_r):

$$dx^1 = \frac{r}{2R_+}dr \qquad dy^1 = R_-\theta^1 \tag{3.38}$$
$$dx^2 = -R_+\theta^1 \qquad dy^2 = \frac{r}{2R_-}dr \tag{3.39}$$
$$dx^3 = -R_+\theta^2 \qquad dy^3 = -R_-\theta^5 \tag{3.40}$$
$$dx^4 = -R_+\theta^3 \qquad dy^4 = -R_-\theta^6 \tag{3.41}$$
$$dx^5 = -R_+\theta^4 \qquad dy^5 = -R_-\theta^7 \tag{3.42}$$

Using equations 3.38-3.42 we obtain:

$$dz^1 = \frac{r}{2R_+}dr + iR_-\theta^1 \tag{3.43}$$
$$dz^2 = -R_+\theta^1 + i\frac{r}{2R_-}dr \tag{3.44}$$
$$dz^3 = -R_+\theta^2 - iR_-\theta^5 \tag{3.45}$$
$$dz^4 = -R_+\theta^3 - iR_-\theta^6 \tag{3.46}$$
$$dz^5 = -R_+\theta^4 - iR_-\theta^7 \tag{3.47}$$

As discussed in section 2, the extendability of a particular linear combination of tensor products of θ^i and X^i is a problem in representation theory. In particular, a tensor extends to an SO(5)-invariant tensor field if and only if it is stabilised by $Ad_{SO(3)}$. Most of the expressions 3.38-3.42 and 3.43-3.47 are thus only valid at p_r. In what follows, we can luckily entirely avoid the problem of classifying extendable tensors. Because of our setup, all abstract tensors we will be working with will be SO(5)-invariant by default. We will thus always attempt to find their expression at the point p_r and it will necessarily be the case that the answer we get is an $Ad_{SO(3)}$-invariant element. The formulae 3.43-3.47 will be particularly useful in this endeavour. All complex geometric calculations we will be required to carry out are easily performed in the $z, \overline{z}$ coordinates. The forumale 3.43-3.47 will then translate the results in the language of invariant forms.

3.2 SO(5)-Invariant Kähler Structures

We now turn to the problem of finding SO(5)-invariant Kähler structures on X^8. Since the second cohomology group vanishes, any Kähler structure comes from a global Kähler potential. We will compute general formulae for SO(5)-invariant Kähler structures coming from SO(5)-invariant Kähler potentials in terms of invariant forms.

3.2.1 The Kähler Form

We look for a Kähler form on X^8 with global Kähler potential:

$$\mathcal{F}(r^2)$$

The resulting form will then be:

$$\omega \in \Lambda^{1,1}T^\star M$$

$$\omega = \frac{i}{2}\partial\overline{\partial}\mathcal{F}(r^2) \tag{3.48}$$

Note that ω is then automatically SO(5)-invariant since for $g \in SO(5)$ we have:

$$g^\star\omega = g^\star\frac{i}{2}\partial\overline{\partial}\mathcal{F}(r^2) = \frac{i}{2}\partial\overline{\partial}\left(\mathcal{F}(r^2)\circ g\right) = \frac{i}{2}\partial\overline{\partial}\mathcal{F}(r^2) = \omega$$

We now use the formulae from the previous section to write $\omega_{|_{p_r}}$ in terms of the potential $\mathcal{F}(r^2)$ and invariant combinations of the θ^i. We calculate:

$$\begin{aligned}\omega &= \frac{i}{2}\partial\overline{\partial}\mathcal{F}(r^2) = \frac{i}{2}\partial\left(\mathcal{F}'(r^2)\overline{\partial}r^2\right)\\ &= \frac{i}{2}\left(\mathcal{F}''(r^2)\partial r^2\wedge\overline{\partial}r^2 + \mathcal{F}'(r^2)\partial\overline{\partial}r^2\right)\\ &= \frac{i}{2}\mathcal{F}'(r^2)\sum_{j=1}^{5}dz^j\wedge d\overline{z}^j + \frac{i}{2}\mathcal{F}''(r^2)\sum_{j=1}^{5}\overline{z^j}dz^j\wedge\sum_{j=1}^{5}z^j d\overline{z^j}\end{aligned} \tag{3.49}$$

where in the last step we have explicitly calculated ∂r^2, $\overline{\partial}r^2$ and $\partial\overline{\partial}r^2$ using the standard coordinates in $\mathbb{C}^5$. For this calculation, it is useful to write:

$$r^2 = \sum_{j=1}^{5}z^j\overline{z}^j$$

We now substitute 3.43-3.47 into 3.49 to obtain:

$$\omega = P(r)dr\wedge\theta^1 + Q(r)\left(\theta^{25}+\theta^{36}+\theta^{47}\right) \tag{3.50}$$

where we have introduced the functions:

$$P(r) \stackrel{\text{def}}{=} \frac{r}{2}\left(\frac{R_+}{R_-}+\frac{R_-}{R_+}\right)\mathcal{F}'(r^2) + 2rR_+R_-\mathcal{F}''(r^2) \tag{3.51}$$

$$Q(r) \stackrel{\text{def}}{=} R_+R_-\mathcal{F}'(r^2) \tag{3.52}$$

Direct calculation shows that the volume form associated to the Kähler structure defined by ω is given by:

$$\mathrm{Vol}_\omega = \frac{\omega^4}{4!} = -PQ^3 dr\wedge\theta^{1234567} \tag{3.53}$$

3.2.2 The Complex Structure and the Kähler Metric

Having expressed the Kähler form in terms of invariant forms and the invariant potential (formula in 3.50), we now write down the complex structure J in this language and proceed to write down an expression for the associated Kähler metric.

Recall that X^8 is a complex submanifold of $\mathbb{C}^5$. As such, the complex structure J is induced from the standard complex structure of the ambient space given by:

$$\partial_{x^j}\mapsto\partial_{y^j}, \quad \partial_{y^j}\mapsto -\partial_{x^j} \tag{3.54}$$

Using equations 3.30-3.22, we discover:

$$JX_1 = -\frac{2R_+R_-}{r}\partial_r \qquad J\partial_r = \frac{r}{2R_+R_-}X_1 \tag{3.55}$$

$$JX_2 = \frac{R_+}{R_-}X_5 \qquad JX_5 = -\frac{R_-}{R_+}X_2 \tag{3.56}$$

$$JX_3 = \frac{R_+}{R_-}X_6 \qquad JX_6 = -\frac{R_-}{R_+}X_3 \tag{3.57}$$

$$JX_4 = \frac{R_+}{R_-}X_7 \qquad JX_7 = -\frac{R_-}{R_+}X_4 \tag{3.58}$$

$$\tag{3.59}$$

The associated metric is given by:

$$g(\cdot,\cdot) = \omega(\cdot, J\cdot)$$

Using this formula in conjunction with 3.50 and 3.55-3.58 we see that all the off-diagonal components of the metric vanish and we have:

$$g(X_1, X_1) = \frac{2R_+R_-}{r}P, \quad g(\partial_r, \partial_r) = \frac{r}{2R_+R_-}P \tag{3.60}$$

$$g(X_2, X_2) = g(X_3, X_3) = g(X_4, X_4) = \frac{R_+}{R_-}Q \tag{3.61}$$

$$g(X_5, X_5) = g(X_6, X_6) = g(X_7, X_7) = \frac{R_-}{R_+}Q \tag{3.62}$$

We hence obtain the following expression for the metric in terms of invariant forms:

$$\begin{aligned} g = {} & \frac{rP}{2R_+R_-}dr \otimes dr + \frac{2R_+R_-P}{r}\theta^1 \otimes \theta^1 \\ & + \frac{R_+Q}{R_-}\left(\theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3 + \theta^4 \otimes \theta^4\right) \\ & + \frac{R_-Q}{R_+}\left(\theta^5 \otimes \theta^5 + \theta^6 \otimes \theta^6 + \theta^7 \otimes \theta^7\right) \end{aligned} \tag{3.63}$$

3.3 The SO(5)-Invariant Holomorphic Volume Form

We now turn our attention to the canonical bundle of X^8. We first prove it is trivial by constructing an explicit holomorphic trivialization. We then derive a formula for this trivialization in terms of invariant forms.

Proposition 3.5. *The bundle K_{X^8} is holomorphically trivial.*

Proof. Let $S_i \subset \mathbb{C}^5$ be the open subset where $z_i \neq 0$. Introduce the following $(n,0)$-form on S_i:

$$\Omega_i \stackrel{\text{def}}{=} \frac{1}{z^i}dz^{i+1} \wedge dz^{i+2} \wedge ... \wedge dz^{i-1} \tag{3.64}$$

where the indices in 3.64 are reduced mod 5.

We claim that the forms $\iota^\star_{X^8}\Omega_i$ glue to a global holomorphic volume form on X^8. Indeed, every point $p \in X^8$ has a non-vanishing coordinate. It follows that at least one of the Ω_i is well defined and non-vanishing at p. Furthermore, if the forms glue, the result is definitely holomorphic as it is locally the pullback of a holomorphic form by a holomorphic function. It therefore remains to prove that for every $p \in X^8$, all the Ω_i that are defined at p agree on T_pX^8.

Fix $p \in X^8$. Without loss of generality we assume that $p_5 \neq 0$. Recall the expression 3.21 for the inclusion of the tangent space T_pX^8 in $\mathbb{C}^5$. From this it follows that:

$$T_pX^8 = \text{Span}_{\mathbb{C}}\left(v_1, ..., v_4\right) \tag{3.65}$$

where:

$$v_i = \partial_{x^i} - \frac{p_i}{p_5}\partial x^5 \tag{3.66}$$

Denote the dual basis by $\beta^1, ..., \beta^4$. We then have:

$$\Lambda^{4,0}T^\star_pX^8 = \Lambda^4_{\mathbb{C}}T^\star_pX^8 = \text{Span}_{\mathbb{C}}\left(\beta^1 \wedge ... \wedge \beta^4\right)$$

It the suffices to check that for any $i = 1, ..., 4$ such that $p_i \neq 0$ we have:

$$\Omega_i(v_1, ..., v_4) = \Omega_5(v_1, ..., v_4)$$

We begin by computing:

$$\Omega_5(v_1, ..., v_4) = \frac{1}{p_5} dz^1 \wedge ... \wedge dz^4(v_1, ..., v_4) = \frac{1}{p_5}$$

where we have observed that when we expand out the wedge product in terms of indecomposable tensors, only the $dz^1 \otimes ... \otimes dz^4$ term gives a non-zero answer.

If all the other p_i vanish we are done. Otherwise $p_i \neq 0$ for some i. For the sake of demonstrating how the computation works we assume $p_1 \neq 0$ and work with p_1. The calculation for the other indices is almost identical. We compute:

$$\begin{aligned}\Omega_1(v_1, ..., v_4) &= \frac{1}{p_1} dz^2 \wedge ... \wedge dz^5(v_1, ..., v_4) \\ &= -\frac{1}{p_1}(-1)\frac{p_1}{p_5} = \frac{1}{p_5}\end{aligned}$$

Here, we have observed that when we expand out the wedge product, the only term that survives is $dz^5 \otimes dz^2 \otimes dz^3 \otimes dz^4$. To see this, argue as follows. Each vector v_i evaluates to 0 unless paired with dz^i or dz^5. Unless dz^5 is paired with v_1, the covector paired with v_1 will evaluate to zero. Since dz^5 is no longer available, each of the remaining v_i must be paired with dz^i. Hence the claim. □

Our next task is to derive an expression of the holomorphic volume form Ω in terms of invariant forms. As always, we compute Ω at the point p_r. The expression we obtain will be $Ad_{SO(3)}$-invariant demonstrating that Ω is SO(5)-invariant.

The first coordinate of p_r does not vanish and hence we have:

$$\Omega = \Omega_1 = \frac{1}{R_+} dz^2 \wedge ... \wedge dz^5$$

We use the formulae 3.43-3.47. After a lengthy calculation we discover that:

$$\begin{aligned}\mathfrak{Re}(\Omega) = &R_+^3 \theta^{1234} - R_+ R_-^2 \left(\theta^{1267} + \theta^{1537} + \theta^{1564}\right) \\ &+ \frac{r}{2} dr \wedge \left(R_+ \left(\theta^{237} + \theta^{264} + \theta^{534}\right) - \frac{R_-^2}{R_+}\theta^{567}\right)\end{aligned} \tag{3.67}$$

$$\begin{aligned}\mathfrak{Im}(\Omega) = &- R_-^3 \theta^{1567} + R_+^2 R_- \left(\theta^{1237} + \theta^{1264} + \theta^{1534}\right) \\ &+ \frac{r}{2} dr \wedge \left(R_- \left(\theta^{267} + \theta^{537} + \theta^{564}\right) - \frac{R_+^2}{R_-}\theta^{234}\right)\end{aligned} \tag{3.68}$$

Note that the SO(5)-symmetry of Ω is apparent since every term in 3.67 and 3.68 is $Ad_{SO(3)}$-invariant. This can be checked explicitly by using the decomposition 3.25, for instance.

Finally, we calculate the volume form associated to Ω. We first compute:

$$\Omega \wedge \overline{\Omega} = \big(\mathfrak{Re}(\Omega) + i\mathfrak{Im}(\Omega)\big) \wedge \big(\mathfrak{Re}(\Omega) - i\mathfrak{Im}(\Omega)\big) = \mathfrak{Re}(\Omega) \wedge \mathfrak{Re}(\Omega) + \mathfrak{Im}(\Omega) \wedge \mathfrak{Im}(\Omega) \tag{3.69}$$

We then use 3.67 and 3.68 to see that:

$$\Omega \wedge \overline{\Omega} = -8rR_+^2 R_-^+ dr \wedge \theta^{1234567} \tag{3.70}$$

We can now easily compute:

$$\mathrm{Vol}_\Omega = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega} = -\frac{r}{2} R_+^2 R_-^2 dr \wedge \theta^{1234567} \tag{3.71}$$

3.4 The Stenzel Calabi-Yau Structure

We now impose volume compatibility in the SO(5)-invariant Kähler structure of the previous section to obtain an SO(5)-invariant CY-4 structure. Volume compatibility boils down to a Monge-Ampère equation for $\mathcal{F}(r^2)$ with right hand side determined by Ω. In SO(5)-symmetry this reduces to an ODE. We begin by deriving the ODE and obtaining the solution explicitly. The Calabi-Yau metric obtained through this process is known as the Stenzel metric (Stenzel [11], Oliveira [8]). Having obtained an explicit formula for the Kähler potential associated to the Stenzel metric, we revisit the results of the previous section and examine what they look like for this choice of $\mathcal{F}$. We conclude the section by writing down the formula for the associated Cayley calibration Φ giving rise to the induced Spin(7)-structure.

3.4.1 Solving the Monge-Ampère Equation in SO(5)-Symmetry

We equate the volume forms associated to ω and Ω: i.e. we set:

$$\mathrm{Vol}_\omega = \mathrm{Vol}_\Omega \tag{3.72}$$

Using 3.53 and 3.71, we see that this is equivalent to the ODE:

$$PQ^3 = \frac{r}{2} R_+^2 R_-^2 \tag{3.73}$$

Unpacking the definitions of P (3.51) and Q (3.52), translates the equation to:

$$1 = r^2 \mathcal{F}'(r^2)^4 + (r^4 - 1)\mathcal{F}'(r^2)^3 \mathcal{F}''(r^2) \tag{3.74}$$

We thus obtain a second order nonlinear ODE for $\mathcal{F}$. Observe that the metric only depends on $\mathcal{F}'$. This motivates us to introduce:

$$\mathcal{G}(r^2) \overset{\text{def}}{=} \mathcal{F}'(r^2)^4 \tag{3.75}$$

Writing 3.74 in terms of $\mathcal{G}$ we obtain:

$$1 = r^2 \mathcal{G}(r^2) + \frac{(r^4 - 1)}{4} \mathcal{G}'(r^2) \tag{3.76}$$

We thus reduce the equation to a first order linear ODE for $\mathcal{F}'$. This is soluble by hand using the integrating factor technique. We write $u = r^2$ and multiply the equation by $4(u^2 - 1)$ to obtain:

$$\begin{aligned}
&(u^2-1)^2 \frac{d\mathcal{G}}{du} + 4u(u^2-1)\mathcal{G}(u) = 4(u^2-1) \\
\Rightarrow &\frac{d}{du}\Big((u^2-1)^2 \mathcal{G}(u)\Big) = 4(u^2-1) \\
\Rightarrow &(u^2-1)^2 \mathcal{G}(u) = \frac{4}{3}u^3 - 4u + C \\
\Rightarrow &\mathcal{G}(u) = \frac{4}{3}\frac{u^3 - 3u + C}{(u^2-1)^2}
\end{aligned} \tag{3.77}$$

We therefore have the solution:

$$\mathcal{F}'(r^2) = \left(\frac{4}{3}\right)^{\frac{1}{4}} \left(\frac{r^6 - 3r^2 + C}{(r^4-1)^2}\right)^{\frac{1}{4}} \tag{3.78}$$

We would like to select the constant so that $\mathcal{F}'$ extends continuously at $r^2 = 1$. This forces us to consider $C = 2$, so that the numerator of the fraction vanishes at $r^2 = 1$. We obtain:

$$\mathcal{F}'(r^2) = \left(\frac{4}{3}\right)^{\frac{1}{4}} \frac{\left(r^2+2\right)^{\frac{1}{4}}}{(r^2+1)^{\frac{1}{2}}} \tag{3.79}$$

It is then obvious that $\mathcal{F}'$ extends smoothly to $r^2 = 1$.

Our task is to write down the functions P and Q in terms of r. The relation 3.74 gives:

$$\mathcal{F}''(r^2) = \frac{1 - r^2\mathcal{F}'(r^2)^4}{(r^4-1)\mathcal{F}'(r^2)^3} \tag{3.80}$$

Combining this with the relation 3.51 we obtain:

$$P(r) = \frac{r}{2}\left(\frac{R_+}{R_-} + \frac{R_-}{R_+}\right)\mathcal{F}'(r^2) + 2rR_+R_-\frac{1 - r^2\mathcal{F}'(r^2)^4}{(r^4-1)\mathcal{F}'(r^2)^3} \tag{3.81}$$

A short calculation gives:

$$P(r) = \frac{r}{2R_+R_-\mathcal{F}'(r^2)^3} \tag{3.82}$$

Incorporating 3.79 we obtain:

$$P(r) = \left(\frac{3}{4}\right)^{\frac{3}{4}} \frac{r(r^2+1)}{(r^2+2)^{\frac{3}{4}}(r+1)^{\frac{1}{2}}(r-1)^{\frac{1}{2}}} \tag{3.83}$$

Similarly, we determine $Q(r)$. Using 3.52 we obtain:

$$Q(r) = \frac{1}{2}\left(\frac{4}{3}\right)^{\frac{1}{4}}(r^2+2)^{\frac{1}{4}}(r+1)^{\frac{1}{2}}(r-1)^{\frac{1}{2}} \tag{3.84}$$

Note that as $r \to 1$, we have that $P(r) \to \infty$ monotonically. This is merely a coordinate singularity. This is clear, for instance, by recalling that ω is obtained on X^8 by a global smooth Kähler potential and is thus smooth everywhere.

3.4.2 Some Remarks on the Geometry

We now study how the lengths of the basis vectors at p_r vary with r. Using 3.63 we observe the following results:

$$|X_1|^2 = \left(\frac{3}{4}\right)^{\frac{3}{4}} \frac{(r^2+1)^{\frac{3}{2}}}{(r^2+2)^{\frac{3}{4}}} \tag{3.85}$$

$$|\partial_r|^2 = \left(\frac{3}{4}\right)^{\frac{3}{4}} \frac{r(r^2+1)}{(r+1)^{\frac{1}{2}}(r-1)^{\frac{1}{2}}} \tag{3.86}$$

$$|X_2|^2 = |X_3|^2 = |X_4|^2 = \frac{1}{2}\left(\frac{4}{3}\right)^{\frac{1}{4}}(r^2+1)^{\frac{1}{2}}(r^2+2)^{\frac{1}{4}} \tag{3.87}$$

$$|X_6|^2 = |X_7|^2 = |X_8|^2 = \frac{1}{2}\left(\frac{4}{3}\right)^{\frac{1}{4}} \frac{(r^2+2)^{\frac{1}{4}}(r+1)(r-1)}{(r^2+1)^{\frac{1}{2}}} \tag{3.88}$$

We observe that as $r \to 1$, $|\partial_r|^2$ blows up monotonically, $|X_1|^2$, $|X_2|^2$, $|X_3|^2$ and $|X_4|^2$ approach 1 and $|X_5|^2, |X_6|^2, |X_7|^2$ tend to 0. Recall that over the singular orbit, the kernel of the projection map 3.29 extends to $\mathfrak{so}(4)$ and X_5, X_6, X_7 project to 0. Consequently, the decay of their norms as $r \to 1$ is a property true of any smooth metric on $T^\star S^4$.

The SO(4) orbit of p_r is the round 3-sphere S^3_ρ of radius:

$$\rho^2 = \frac{1}{2}\left(\frac{4}{3}\right)^{\frac{1}{4}} \frac{(r^2+2)^{\frac{1}{4}}(r+1)(r-1)}{(r^2+1)^{\frac{1}{2}}}$$

The 3-dimensional volume of S^3_ρ is given by:

$$\mathrm{Vol}\left(S^3_\rho\right) = 2\pi^2\rho^3 = \frac{2^{\frac{1}{4}}\pi^2}{3^{\frac{3}{8}}} \frac{(r^2+2)^{\frac{3}{8}}(r+1)^{\frac{3}{2}}(r-1)^{\frac{3}{2}}}{(r^2+2)^{\frac{3}{4}}}$$

Since SO(5) acts isometrically, the same is true of all the fibers of the sphere bundle corresponding to a given value of r.

The singular orbit is the round unit radius 4-sphere S^4_1 with 4-dimensional volume equal to $\frac{8\pi^2}{3}$.

3.4.3 The Cayley Calibration

As discussed in section 1, an SU(4) structure on a vector space induces a Cayley calibration. Consequently, an SU(4)- structure over a smooth manifold induces a Cayley calibration over each tangent space. In this way, we obtain a Spin(7) structure. If the SU(4)-structure is torsion free (i.e. the Kähler condition is satisfied and consequently ω is a Calabi-Yau metric), the Cayley calibration is closed. Therefore, the induced Spin(7) structure is also torsion free. We conclude that a CY 4-fold is in a natural way a Spin(7) manifold.

The induced Cayley calibration on a CY 4-fold can be written in terms of the Kähler form and the holomorphic volume form as follows:

$$\Phi = \frac{\omega^2}{2} + \mathfrak{Re}(\Omega) \tag{3.89}$$

Using the formula 3.50 for the Kähler form in terms of invariant forms we can compute that:

$$\omega^2 = 2PQdr \wedge \left(\theta^{125} + \theta^{136} + \theta^{147}\right) + 2Q^2 \left(\theta^{2536} + \theta^{2547} + \theta^{3647}\right) \tag{3.90}$$

Combining this with the formula 3.67 for the real part of the holomorphic volume form and using 3.89, we obtain:

$$\begin{aligned}\Phi = dr \wedge &\left[PQ\left(\theta^{125} + \theta^{136} + \theta^{147}\right) + \frac{rR_+}{2}\left(\theta^{237} + \theta^{264} + \theta^{534}\right) - \frac{R_-^2}{R_+}\theta^{567}\right] \\ &+ R_+^3\theta^{1234} - R_+R_-^2\left(\theta^{1267} + \theta^{1537} + \theta^{1564}\right) + Q^2\left(\theta^{2536} + \theta^{2547} + \theta^{3647}\right)\end{aligned} \tag{3.91}$$

We immediately make the following observation. When we pull back Φ to the singular S^4 by the inclusion map, only the θ^{1234} term survives. Furthermore, on S^4 we have $r = 1$. We therefore get:

$$\iota_{S^4}^{\star}\Phi = \theta^{1234} \tag{3.92}$$

We conclude that the singluar orbit is calibrated for Φ and is therefore a Cayley submanifold of the Spin(7) manifold (X^8, Φ). As such, it is volume minimizing in its homology class (Joyce [2]).

4 SO(5)-Invariant U(1)-Instantons on the Stenzel Manifold

We now have all the ingredients required to study invariant instantons on $\left(T^\star S^4, J, \omega, \Omega\right)$. Naturally, we begin with the simplest setting where the structure group is equal to the unitary group U(1). We first use the results of section 2 to classify SO(5)-invariant U(1) bundles and connections. We then proceed to derive the ODEs describing the evolution of invariant Spin(7) instantons and invariant HYM connections. Finally, we study the ODEs obtained in the previous step and attempt to understand whether or not the solutions extend over the singular orbit.

4.1 SO(5)-Invariant U(1)-Bundles and SO(5)-invariant U(1)-Connections

Let $r > 1$. The SO(5)-invariant U(1) bundles over the orbit:

$$O_r \cong \frac{SO(5)}{SO(3)}$$

correspond to Lie group homomorphisms:

$$\lambda : \mathrm{U}(1) \to \mathrm{SO}(3) \tag{4.1}$$

The derived subalgebra $\mathcal{D}(\mathfrak{so}(3))$ is equal to $\mathfrak{so}(3)$. Since SO(3) is compact, its derived subgroup is closed and we have:

$$\mathrm{Lie}\left([SO(3), SO(3)]\right) = \mathcal{D}(\mathfrak{so}(3)) = \mathfrak{so}(3) \tag{4.2}$$

Since SO(3) is connected, it is generated by the image of its Lie algebra under the exponential map. The latter is contained in the derived subgroup. We conclude that the derived subgroup is equal to the full group.

Since U(1) is abelian, morphisms to U(1) map the derived subgroup to the identity. Hence the only group homomorphism λ as in 4.1 is the trivial map $\phi = 1$. We conclude that the only SO(5) invariant U(1) bundle over $\frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}$ is the trivial bundle:

$$P_1 = \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)} \times \mathrm{U}(1) \tag{4.3}$$

We may pull this bundle back to $T^\star S^4 - S^4$ under the map:

$$T^\star S^4 - S^4 \xrightarrow{\sim} (1, \infty) \times \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)} \twoheadrightarrow \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)} \tag{4.4}$$

This gives use the trivial U(1) bundle over $T^\star S^4 - S^4$:

$$P = \left(T^\star S^4 - S^4\right) \times \mathrm{U}(1) \tag{4.5}$$

P clearly extends smoothly to $T^\star S^4$ as the trivial U(1) bundle, but for now we are going to restrict ourselves to $r > 1$.

SO(5)- invariant U(1)-connections are then parameterised by representation morphisms:

$$\Lambda : \left(\mathfrak{m}, \mathrm{Ad}_{\mathrm{SO}(3)}\right) \to \left(\mathfrak{u}(1), \mathrm{Ad}_{\lambda(\mathrm{SO}(3))}\right) = (i\mathbb{R}, 1) \tag{4.6}$$

Recalling the decomposition 3.27 and applying Schur's lemma, we obtain that:

$$\mathrm{Hom}_{\mathrm{SO}(3)}\left(\mathfrak{m}, \mathfrak{u}(1)\right) = i\mathbb{R} \tag{4.7}$$

where the imaginary number $i\alpha$ corresponds to:

$$\Lambda_\alpha \stackrel{\mathrm{def}}{=} i\alpha\theta^1 \tag{4.8}$$

We see that a general SO(5)-invariant connection over $T^\star S^4 - S^4$ can be written as:

$$A = i\alpha(r)\theta^1 \tag{4.9}$$

The curvature of A is then:

$$\begin{aligned} F_A &= dA \\ &= i\frac{d\alpha}{dr}dr \wedge \theta^1 + i\alpha(r)d\theta^1 \end{aligned} \tag{4.10}$$

To compute the second term we use the Mauer-Cartan relations (Kobayashi, Nomizu [3] p. 41):

$$d\theta^k = -\frac{1}{2}c^k_{\mu\nu}\theta^{\mu\nu} \tag{4.11}$$

where $c^k_{\mu\nu}$ are the structure constants of $\mathfrak{so}(5)$. The structure constants can be computed using (3.23) and (3.24). Carrying out the calculation gives:

$$d\theta^1 = \theta^{25} + \theta^{36} + \theta^{47} \tag{4.12}$$

Incorporating this into (4.10) we obtain:

$$F_A = i\frac{d\alpha}{dr}dr \wedge \theta^1 + i\alpha(r)\left(\theta^{25} + \theta^{36} + \theta^{47}\right) \tag{4.13}$$

4.2 The SO(5)-Invariant Spin(7) Instanton ODE

The Spin(7) instanton equation reads:

$$\star_g F_A = -\Phi \wedge F_A \tag{4.14}$$

We use the formulae obtained in the previous sections to express each side in terms of invariant forms. We work on $T_{p_r}X^8$ with the X_i frame. Since the metric diagonalises we have:

$$\star_g \theta^{i_1} \wedge ... \wedge \theta^{i_k} = \frac{\sqrt{\det(g)}}{g_{i_1 i_1}\cdots g_{i_k i_k}}\theta^{i_{k+1}} \wedge ... \wedge \theta^{i_n}$$

where $i_1, ..., i_n$ is an even permutation of $1, ..., n$. Using (4.15) and (3.63) we obtain the results:

$$\begin{aligned} \star_g dr \wedge \theta^1 &= -\frac{Q^3}{A}\theta^{234567} && (4.15) \\ \star_g \theta^{25} &= -PQdr \wedge \theta^{13467} && (4.16) \\ \star_g \theta^{36} &= -PQdr \wedge \theta^{12356} && (4.17) \\ & && (4.18) \end{aligned}$$

Using these expressions we obtain:

$$\star_g F_A = -i\frac{Q^3}{P}\frac{d\alpha}{dr}\theta^{234567} - iPQdr \wedge \left(\theta^{13467} + \theta^{12457} + \theta^{12356}\right) \tag{4.19}$$

We now use (3.91) and (4.13) to compute:

$$\Phi \wedge F_A = -3iQ^2\alpha(r)\theta^{234567} - i\left(Q^2\frac{d\alpha}{dr} + 2PQ\alpha(r)\right)dr \wedge \left(\theta^{13467} + \theta^{12457} + \theta^{12356}\right) \tag{4.20}$$

Imposing 4.14 and comparing coefficients gives two equations. These are the same and read:

$$\frac{d\alpha}{dr} = -3\frac{P}{Q}\alpha \tag{4.21}$$

4.3 The SO(5)-Invariant HYM ODE

The Hermitian Yang-Mills equations read:

$$F_A \wedge \star\omega = 0 \tag{4.22}$$
$$F_A \wedge \Omega = 0 \tag{4.23}$$

The latter statement holds identically. This can be seen by direct computation using (3.67), (3.68) and (4.13).

It follows that an SO(5)-invariant U(1) connection A is HYM if and only if 1.47 holds.

Over a Hermitian manifold of complex dimension n, we have:

$$\star_g \omega = \frac{\omega^{n-1}}{(n-1)!} \tag{4.24}$$

where g is the Kähler metric associated to ω by the complex structure. Using (3.50) and (3.90) we compute:

$$\omega^3 = 6PQ^2 dr \wedge \left(\theta^{12536} + \theta^{12547} + \theta^{13647}\right) + 6B^3\theta^{253647} \tag{4.25}$$

Using (4.24), (4.25) and (4.13) we calculate:

$$\begin{aligned} F_A \wedge \star\omega &= F_A \wedge \frac{\omega^3}{3!} \\ &= -i\left(Q^3\frac{d\alpha}{dr} + 3PQ^2\alpha(r)\right) dr \wedge \theta^{1234567} \end{aligned} \tag{4.26}$$

It follows that an SO(5)-invariant U(1)-connection is HYM if and only if:

$$\frac{d\alpha}{dr} = -3\frac{P}{Q}\alpha \tag{4.27}$$

We observe that this equation is the same as (4.21).

The uniqueness theorem for solutions to ODEs with a Lipschitz vector field implies that:

Theorem 4.1. *An SO(5)-invariant U(1)-connection over $T^\star S^4 - S^4$ equipped with the Stenzel Calabi-Yau structure is a Spin(7) instanton if and only if it is Hermitian-Yang-Mills.*

4.4 Breakdown Near the Singular Orbit

We study the ODE (4.27). Using (3.83) and (3.84) we write it as:

$$\frac{da}{dr} = -\frac{9}{2}\frac{r(r^2+1)}{(r^2+2)(r+1)(r-1)}\alpha(r) \tag{4.28}$$

We integrate (4.28) directly to see that the solution takes the following form for some $K \in \mathbb{R}$:

$$\alpha(r) = \frac{K}{(r^2+2)^{\frac{3}{4}}(r+1)^{\frac{3}{2}}(r-1)^{\frac{3}{2}}} \tag{4.29}$$

The derivative of this function can be computed using (4.29) and (4.28). We obtain:

$$\frac{da}{dr} = -\frac{9K}{2}\frac{r(r^2+1)}{(r^2+2)^{\frac{7}{4}}(r+1)^{\frac{5}{2}}(r-1)^{\frac{5}{2}}} \tag{4.30}$$

Recalling the formulae (4.9) and (4.13) and incorporating (4.29) and (4.30), we formulate the following theorem:

Theorem 4.2. *Let $M = T^\star S^4 - S^4$ be equipped with the Stenzel Calabi-Yau structure $(\omega_S, \Omega_S, J_S)$. Let P be the unique homogeneous U(1) bundle over M (i.e. the trivial bundle). There exists a unique (up to a scalar multiple determined by $K \in \mathbb{R}$) smooth SO(5)-invariant Spin(7) instanton $A_{Spin(7)} \in \mathcal{A}(P)$ given by:*

$$A_{Spin(7)} = \frac{iK}{(r^2+2)^{\frac{3}{4}}(r+1)^{\frac{3}{2}}(r-1)^{\frac{3}{2}}}\theta^1 \tag{4.31}$$

The curvature of $A_{Spin(7)}$ is given by:

$$F_{A_{Spin(7)}} = iK\left(-\frac{9}{2}\frac{r(r^2+1)}{(r^2+2)^{\frac{7}{4}}(r+1)^{\frac{5}{2}}(r-1)^{\frac{5}{2}}}dr\wedge\theta^1 + \frac{\theta^{25}+\theta^{36}+\theta^{47}}{(r^2+2)^{\frac{3}{4}}(r+1)^{\frac{3}{2}}(r-1)^{\frac{3}{2}}}\right) \tag{4.32}$$

Recall that the norm of the invariant 1-form θ^1 is a function of r that stays finite and in fact converges to 1 as $r \to 1$. It is then evident that the pointwise norm of the invariant Spin(7) instanton A blows up as $r \to 1$. Since the metric extends smoothly to the singular orbit, this behaviour is precluded for connections that are continuous over the whole space. This provides the following result:

Theorem 4.3. *There exist no SO(5)-invariant Spin(7) instantons (and therefore also HYM connections) on $\left(T^\star S^4, \omega_S, \Omega_S, J_S\right)$ apart from the trivial connection $A = 0$ (corresponding to $K = 0$).*

As a closing remark, we note that blowup around Cayley 4-folds is an interesting feature of the Spin(7)-instanton equation. It is related to the non-compactness of the moduli space. In Donaldson theory, noncompactness occurs in the form of a sequence of ASD instantons failing to have a limit due to finitely many point singularities. In the 8-dimensional Spin(7) setting points are typically replaced by four-dimensional Cayley submanifolds.

The ideas discussed in this report are work in progress. We plan to study more complicated structure groups and other cohomogeneity one spaces in the near future. The next step will be to study the structure group SO(3) on the Stenzel manifold. We believe the non-existence result we encountered has to do with abelian gauge theory being too coarse to capture the behaviour we would like to see. We hope to construct a Spin(7) instanton that is not HYM and smoothly extends over the singular orbit. For twisted homogeneous bundles, the question of extendibility is more subtle. Eschenburg and Wang (Eschenburg-Wang [1]) have devised a technique to solve the extendibility problem for general geometric sturctures in cohomgeneity one. We plan to use their method.

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